

# $\mathbf{GL}(2, \mathbb{R})$ GEOMETRY OF ODE'S

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**ABSTRACT.** We study five dimensional geometries associated with the 5-dimensional irreducible representation of  $\mathbf{GL}(2, \mathbb{R})$ . These are special Weyl geometries in signature  $(3, 2)$  having the structure group reduced from  $\mathbf{CO}(3, 2)$  to  $\mathbf{GL}(2, \mathbb{R})$ . The reduction is obtained by means of a conformal class of totally symmetric 3-tensors. Among all  $\mathbf{GL}(2, \mathbb{R})$  geometries we distinguish a subclass which we term ‘nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  geometries’. These define a unique  $\mathfrak{gl}(2, \mathbb{R})$  connection which has totally skew symmetric torsion. This torsion splits onto the  $\mathbf{GL}(2, \mathbb{R})$  irreducible components having respective dimensions 3 and 7.

We prove that on the solution space of every 5th order ODE satisfying certain three nonlinear differential conditions there exists a nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  geometry such that the skew symmetric torsion of its unique  $\mathfrak{gl}(2, \mathbb{R})$  connection is very special. In contrast to an arbitrary nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  geometry, it belongs to the 3-dimensional irreducible representation of  $\mathbf{GL}(2, \mathbb{R})$ . The conditions for the existence of the structure are lower order equivalents of the Doubrov-Wilczynski conditions found recently by Boris Doubrov [7].

We provide nontrivial examples of 5th order ODEs satisfying the three nonlinear differential conditions, which in turn provides examples of inhomogeneous  $\mathbf{GL}(2, \mathbb{R})$  geometries in dimension five, with torsion in  $\mathbb{R}^3$ .

We also outline the theory and the basic properties of  $\mathbf{GL}(2, \mathbb{R})$  geometries associated with  $n$ -dimensional irreducible representations of  $\mathbf{GL}(2, \mathbb{R})$  in  $6 \leq n \leq 9$ . In particular we give conditions for an  $n$ th order ODE to possess this geometry on its solution space.

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## 1. INTRODUCTION

Let us start with an elementary algebraic geometry in  $\mathbb{R}^3$ . Every *point* on a curve  $(1, x, x^2)$  in  $\mathbb{R}^3$  defines a *straight line* passing through the origin in the dual space  $(\mathbb{R}^3)^*$  via the relation:

$$(1.1) \quad \begin{aligned} \theta^0 + 2\theta^1 x + \theta^2 x^2 &= 0 \\ \theta^1 + \theta^2 x &= 0. \end{aligned}$$

Here  $(\theta^0, \theta^1, \theta^2)$  parametrize points of  $(\mathbb{R}^3)^*$ . When moving along the curve  $(1, x, x^2)$  in  $\mathbb{R}^3$ , the corresponding lines in the dual space  $(\mathbb{R}^3)^*$  sweep out a ruled *surface* there, which is the cone

$$(1.2) \quad (\theta^1)^2 - \theta^0 \theta^2 = 0$$

with the tip in the origin. The points  $(\theta^0, \theta^1, \theta^2)$  lying on this cone may be thought as those, and only those, which admit a common root  $x$  for the pair of equations (1.1). A standard method for determining if two polynomials have a common root is to equate to zero their *resultant*. In the case of equations (1.1) the resultant is:

$$R_3 = \det \begin{pmatrix} \theta^0 & 2\theta^1 & \theta^2 & 0 & 0 \\ 0 & \theta^0 & 2\theta^1 & \theta^2 & 0 \\ 0 & 0 & \theta^0 & 2\theta^1 & \theta^2 \\ \theta^1 & \theta^2 & 0 & 0 & 0 \\ 0 & \theta^1 & \theta^2 & 0 & 0 \end{pmatrix}.$$

It vanishes if and only if condition (1.2) holds.

Before passing to  $\mathbb{R}^n$  with general  $n \geq 3$ , it is instructive to repeat the above considerations in the cases of  $n = 4$  and  $n = 5$ .

A *point* on a curve  $(1, x, x^2, x^3)$  in  $\mathbb{R}^4$  defines a *plane* passing through the origin in the dual space  $(\mathbb{R}^4)^*$  via the relation:

$$(1.3) \quad \begin{aligned} \theta^0 + 3\theta^1 x + 3\theta^2 x^2 + \theta^3 x^3 &= 0 \\ \theta^1 + 2\theta^2 x + \theta^3 x^2 &= 0. \end{aligned}$$

Now  $(\theta^0, \theta^1, \theta^2, \theta^3)$  parametrize points of the dual  $(\mathbb{R}^4)^*$  and when moving along the curve  $(1, x, x^2, x^3)$  in  $\mathbb{R}^4$ , the corresponding planes in  $(\mathbb{R}^4)^*$  sweep out a ruled

*hypersurface* there, which is defined by the vanishing of the resultant of the two polynomials defined in (1.3). This is given by

$$(1.4) \quad -3(\theta^1)^2(\theta^2)^2 + 4\theta^0(\theta^2)^3 + 4(\theta^1)^3\theta^3 - 6\theta^0\theta^1\theta^2\theta^3 + (\theta^0)^2(\theta^3)^2 = 0,$$

as can be easily calculated.

For  $n = 5$ , a *point* on a curve  $(1, x, x^2, x^3, x^4)$  in  $\mathbb{R}^5$  defines a *3-plane* passing through the origin in the dual space  $(\mathbb{R}^5)^*$  via the relation:

$$(1.5) \quad \begin{aligned} \theta^0 + 4\theta^1x + 6\theta^2x^2 + 4\theta^3x^3 + \theta^4x^4 &= 0 \\ \theta^1 + 3\theta^2x + 3\theta^3x^2 + \theta^4x^3 &= 0, \end{aligned}$$

where  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$  parametrize points of the dual  $(\mathbb{R}^5)^*$  as before. And now, when moving along the curve  $(1, x, x^2, x^3, x^4)$  in  $\mathbb{R}^5$ , the corresponding 3-planes in  $(\mathbb{R}^4)^*$  sweep out a ruled *hypersurface* there, which is again defined by the vanishing of the resultant of the two polynomials defined in (1.5). The algebraic expression for this hypersurface in terms of the  $\theta$  coordinates is quite complicated:

$$(1.6) \quad \begin{aligned} &-36(\theta^1)^2(\theta^2)^2(\theta^3)^2 + 54\theta^0(\theta^2)^3(\theta^3)^2 + 64(\theta^1)^3(\theta^3)^3 - 108\theta^0\theta^1\theta^2(\theta^3)^3 + \\ &27(\theta^0)^2(\theta^3)^4 + 54(\theta^1)^2(\theta^2)^3\theta^4 - 81\theta^0(\theta^2)^4\theta^4 - 108(\theta^1)^3\theta^2\theta^3\theta^4 + \\ &180\theta^0\theta^1(\theta^2)^2\theta^3\theta^4 + 6\theta^0(\theta^1)^2(\theta^3)^2\theta^4 - 54(\theta^0)^2\theta^2(\theta^3)^2\theta^4 + 27(\theta^1)^4(\theta^4)^2 \\ &-54\theta^0(\theta^1)^2\theta^2(\theta^4)^2 + 18(\theta^0)^2(\theta^2)^2(\theta^4)^2 + 12(\theta^0)^2\theta^1\theta^3(\theta^4)^2 - (\theta^0)^3(\theta^4)^3 = 0, \end{aligned}$$

but easily calculable.

The beauty of the hypersurfaces (1.2), (1.4) and (1.6) consists in this that they are given by means of homogeneous equations, and thus they descend to the corresponding projective spaces. From the point of view of the present paper, even more important is the fact, that they are  $\mathbf{GL}(2, \mathbb{R})$  *invariant*. By this we mean the following:

Consider a real polynomial of  $(n - 1)$ -th degree

$$(1.7) \quad w(x) = \sum_{i=0}^{n-1} \binom{n-1}{i} \theta^i x^i$$

in the real variable  $x$  with real coefficients  $(\theta^0, \theta^1, \dots, \theta^{n-1})$ . The  $n$ -dimensional vector space  $(\mathbb{R}^n)^*$  of such polynomials may be identified with the space of their coefficients. Now, replacing the variable  $x$  by a new variable  $x'$  such that

$$(1.8) \quad x = \frac{\alpha x' + \beta}{\gamma x' + \delta}, \quad \alpha\delta - \beta\gamma \neq 0,$$

we define a new covector  $(\theta'^0, \theta'^1, \dots, \theta'^{n-1})$  which is related to  $(\theta^0, \theta^1, \dots, \theta^{n-1})$  of (1.7) via

$$\sum_{i=0}^{n-1} \binom{n-1}{i} \theta'^i x'^i = (\gamma x' + \delta)^{n-1} w(x).$$

It is obvious that  $\theta' = (\theta'^0, \theta'^1, \dots, \theta'^{n-1})$  is linearly expressible in terms of  $\theta = (\theta^0, \theta^1, \dots, \theta^{n-1})$ :

$$(1.9) \quad \theta' = \theta \cdot \rho_n(a), \quad a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Here  $a$  corresponds to the  $\mathbf{GL}(2, \mathbb{R})$  transformation (1.8), and the map

$$\rho_n : \mathbf{GL}(2, \mathbb{R}) \rightarrow \mathbf{GL}((\mathbb{R}^n)^*) \cong \mathbf{GL}(n, \mathbb{R})$$

defines the *real  $n$ -dimensional irreducible representation* of  $\mathbf{GL}(2, \mathbb{R})$ . For example, if  $n = 2$ , we have  $w(x) = \theta^0 + 2\theta^1 x + \theta^2 x^2$ , and we easily get

$$\begin{pmatrix} \theta'^0 & \theta'^1 & \theta'^2 \end{pmatrix} = \begin{pmatrix} \theta^0 & \theta^1 & \theta^2 \end{pmatrix} \begin{pmatrix} \delta^2 & \gamma\delta & \gamma^2 \\ 2\beta\delta & \alpha\delta + \beta\gamma & 2\alpha\gamma \\ \beta^2 & \alpha\beta & \alpha^2 \end{pmatrix},$$

so that  $\rho_2$  is given by

$$\rho_2 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta^2 & \gamma\delta & \gamma^2 \\ 2\beta\delta & \alpha\delta + \beta\gamma & 2\alpha\gamma \\ \beta^2 & \alpha\beta & \alpha^2 \end{pmatrix}.$$

Now, let us define  $g(\theta, \theta)$ ,  ${}^4I(\theta, \theta, \theta, \theta)$  and  ${}^5I(\theta, \theta, \theta, \theta, \theta)$  by

$$\begin{aligned} (1.10) \quad & g(\theta, \theta) = \text{the left hand side of (1.2)} \\ & {}^4I(\theta, \theta, \theta, \theta) = \text{the left hand side of (1.4)} \\ & {}^5I(\theta, \theta, \theta, \theta, \theta) = \text{the left hand side of (1.6)}. \end{aligned}$$

We will often abbreviate this notation to the respective:  $g(\theta)$ ,  ${}^4I(\theta)$  and  ${}^5I(\theta)$ .

To explain our comment about the  $\mathbf{GL}(2, \mathbb{R})$  invariance of the respective hyper-surfaces  $g(\theta) = 0$ ,  ${}^4I(\theta) = 0$  and  ${}^5I(\theta) = 0$  we calculate  $g(\theta')$ ,  ${}^4I(\theta')$  and  ${}^5I(\theta')$  with  $\theta'$  as in (1.9). The result is

$$\begin{aligned} g(\theta') &= (\alpha\delta - \beta\gamma)^2 g(\theta) \\ {}^4I(\theta') &= (\alpha\delta - \beta\gamma)^4 {}^4I(\theta) \\ {}^5I(\theta') &= (\alpha\delta - \beta\gamma)^6 {}^5I(\theta). \end{aligned}$$

Thus the vanishing of the expressions  $g(\theta)$ ,  ${}^4I(\theta)$  and  ${}^5I(\theta)$  is invariant under the action (1.9) of the irreducible  $\mathbf{GL}(2, \mathbb{R})$  on  $(\mathbb{R}^n)^*$ .

We are now ready to discuss the general case  $n \geq 3$  of the *rational normal curve*  $(1, x, x^2, \dots, x^{n-1})$  in  $\mathbb{R}^n$ . Associated with this curve is a pair of polynomials, namely  $w(x)$  as in (1.7), and its *derivative*  $\frac{dw}{dx}$ . We consider the relation

$$(1.11) \quad w(x) = 0 \quad \& \quad \frac{dw}{dx} = 0.$$

This gives a *correspondence* between the *points* on the curve  $(1, x, x^2, \dots, x^{n-1})$  in  $\mathbb{R}^n$  and the  $(n-2)$ -*planes* passing through the origin in the dual space  $(\mathbb{R}^n)^*$  parametrized by  $(\theta^0, \theta^1, \dots, \theta^{n-1})$ . When moving along the rational normal curve in  $\mathbb{R}^n$ , the corresponding  $(n-2)$ -planes in  $(\mathbb{R}^n)^*$  sweep out a ruled *hypersurface* there. This is defined by the vanishing of the resultant,  $R(w(x), \frac{dw}{dx})$ , of the two polynomials in (1.11). The algebraic expression for this hypersurface is the vanishing of a homogeneous polynomial, let us call it  $I(\theta)$ , of order  $2(n-2)$ , in the coordinates  $(\theta^0, \theta^1, \dots, \theta^{n-1})$ . The hypersurface  $I(\theta) = 0$  in  $(\mathbb{R}^n)^*$  is  $\mathbf{GL}(2, \mathbb{R})$  invariant, since the property of the two polynomials  $w(x)$  and  $\frac{dw}{dx}$  to have a common root is independent of the choice (1.8) of the coordinate  $x$ . Thus  $\mathbf{GL}(2, \mathbb{R})$  is *included* in the stabilizer  $G_I$  of  $I$  under the action of the full  $\mathbf{GL}(n, \mathbb{R})$  group. This stabilizer, by definition, is a subgroup of  $\mathbf{GL}(n, \mathbb{R})$  with elements  $b \in G_I \subset \mathbf{GL}(n, \mathbb{R})$  such that  $I(\theta \cdot b) = (\det b)^{\frac{2(n-2)}{n}} I(\theta)$ . Moreover, in  $n = 4, 5$ , it turns out that  $G_I$  is *precisely* the group  $\mathbf{GL}(2, \mathbb{R})$  in the corresponding irreducible representation  $\rho_n$ . Thus if  $n = 4, 5$  one can characterize the irreducible  $\mathbf{GL}(2, \mathbb{R})$  in  $n$  dimensions as the stabilizer of the polynomial  $I(\theta)$ .

Crucial for the present paper is an observation of Karl Wünschmann that the algebraic geometry and the correspondences we were describing above, naturally appear in the analysis of solutions of the ODE  $y^{(n)} = 0$ . Indeed, following Wünschmann<sup>1</sup> (see the Introduction in his PhD thesis [21], pp. 5-6), we note the following:

Consider the third order ODE:  $y''' = 0$ . Its general solution is  $y = c_0 + 2c_1x + c_2x^2$ , where  $c_0, c_1, c_2$  are the integration constants. Thus, the solution space of the ODE  $y''' = 0$  is  $\mathbb{R}^3$  with solutions identified with points  $\mathbf{c} = (c_0, c_1, c_2) \in \mathbb{R}^3$ . The solutions to the ODE  $y''' = 0$  may be also identified with curves  $y(x) = c_0 + 2c_1x + c_2x^2$ , actually parabolas, in the plane  $(x, y)$ . Suppose now, that we take two solutions of  $y''' = 0$  corresponding to two neighbouring points  $\mathbf{c} = (c_0, c_1, c_2)$  and  $\mathbf{c} + d\mathbf{c} = (c_0 + dc_0, c_1 + dc_1, c_2 + dc_2)$  in  $\mathbb{R}^3$ . Among all pairs of neighbouring solutions we choose only those, which have the property that their corresponding curves  $y = y(x)$  and  $y + dy = y(x) + dy(x)$  touch each other, at some point  $(x_0, y_0)$  in the plane  $(x, y)$ . If we do not require anything more about the properties of this incidence of the two curves, we say that solutions  $\mathbf{c}$  and  $\mathbf{c} + d\mathbf{c}$  have *zero order contact* at  $(x_0, y_0)$ .

In this ‘baby’ example everything is very simple:

To get the criterion for the solutions to have zero order contact we first write the curves  $y = c_0 + 2c_1x + c_2x^2$  and  $y + dy = c_0 + dc_0 + 2(c_1 + dc_1)x + (c_2 + dc_2)x^2$  corresponding to  $\mathbf{c}$  and  $\mathbf{c} + d\mathbf{c}$ . Thus the solutions have zero order contact at  $(x_0, y(x_0))$  provided that  $dy(x_0) = 0$ , i.e. if and only if

$$dc_0 + 2x_0dc_1 + x_0^2dc_2 = 0.$$

This shows that such a contact is possible if and only if the determinant

$$g(d\mathbf{c}, d\mathbf{c}) = (dc_1)^2 - dc_0dc_2$$

is *nonnegative*, since otherwise the quadratic equation for  $x_0$  has no solutions. Unexpectedly, we find that the requirement for the two neighbouring solution curves of  $y''' = 0$  to have zero order contact at some point is equivalent to the requirement that the corresponding two neighbouring points  $\mathbf{c}$  and  $\mathbf{c} + d\mathbf{c}$  in  $\mathbb{R}^3$  are *spacelike separated* in the *Minkowski metric*  $g$  in  $\mathbb{R}^3$ . This is the discovery of Wünschmann that is quoted in Elie Cartan’s 1941 year’s paper<sup>2</sup> [5].

Now we consider the neighbouring solutions  $\mathbf{c}$  and  $\mathbf{c} + d\mathbf{c}$  of  $y''' = 0$  which are *null separated* in the metric  $ds^2$ . What we can say about the corresponding curves in the plane  $(x, y)$ ?

To answer this we need the notion of a *first order contact*: Two neighbouring solution curves  $y = c_0 + 2c_1x + c_2x^2$  and  $y + dy = c_0 + 2c_1x + c_2x^2 + (dc_0 + 2x_0dc_1 + x_0^2dc_2)$  of  $y''' = 0$ , corresponding to  $\mathbf{c}$  and  $\mathbf{c} + d\mathbf{c}$  in  $\mathbb{R}^3$ , have first order contact at  $(x_0, y_0)$  iff they have zero order contact at  $(x_0, y_0)$  and, in addition, their curves of

<sup>1</sup>We are very grateful to Niels Schuman, who found a copy of Wünschmann thesis in the *city* library of Berlin and sent it to us. It was this copy, which after translation from German by Denson Hill, led us to write this introduction.

<sup>2</sup>It is worthwhile to remark, that Wünschmann thesis is dated ‘1905’, the same year in which Einstein published his famous special relativity theory paper [9]. It was not until three years later when Minkowski gave the geometric interpretation of Einstein’s theory in terms of his metric [15]. Perhaps Wünschmann was the first who ever wrote such metric in a scientific paper. This is a very interesting feature of Wünschmann thesis: he calls the expressions like  $(dc_1)^2 - dc_0dc_2 = 0$ , a *Mongesche Gleichung* rather than a *cone in the metric*, because the notion of a metric with signature different than the Riemannian one was not yet abstracted!

first derivatives,  $y' = 2c_1 + 2c_2x$  and  $y' + dy' = 2(c_1 + dc_1) + 2(c_2 + dc_2)x$ , have zero order contact at  $(x_0, y_0)$ . Thus the condition of first order contact at  $(x_0, y(x_0))$  is equivalent to  $dy(x_0) = 0$  and  $dy'(x_0) = 0$ , i.e. to the condition that  $x_0$  is a *simultaneous* root for

$$(1.12) \quad \begin{aligned} dc_0 + 2x_0dc_1 + x_0^2dc_2 &= 0 \\ dc_1 + x_0dc_2 &= 0. \end{aligned}$$

Solving the second of these equations for  $x_0$ , and inserting it into the first, after an obvious simplification, we conclude that  $(dc_1)^2 - dc_0dc_2 = 0$ . Thus we get the interpretation of the *null separated* neighbouring points in  $\mathbb{R}^3$  as the solutions of  $y''' = 0$  whose curves in the  $(x, y)$  plane are neighbouring and have first order contact at some point.

Wünschmann notes that the procedure described here for the equation  $y''' = 0$  can be repeated for the equation  $y^{(n)} = 0$  for arbitrary  $n \geq 3$ . In the cases of  $n = 4$  and  $n = 5$  he however passes to the discussion of the solutions that have contact of order  $(n - 2)$  rather than one. This is an interesting possibility, complementary in a sense to the one in which the solutions have first order contact. Wünschmann spends rest of the thesis studying it. But we will not discuss it here.

Since Wünschmann does not discuss the first order contact of the solutions in  $n = 4, 5$ , let us look closer onto these two cases:

The general solution to  $y^{(4)} = 0$  is  $y = c_0 + 3c_1x + 3c_2x^2 + c_3x^3$ , and the general solution to  $y^{(5)} = 0$  is  $y = c_0 + 4c_1x + 6c_2x^2 + 4c_3x^3 + c_4x^4$ . Thus now the solutions are points  $\mathbf{c}$  in  $\mathbb{R}^4$  and  $\mathbb{R}^5$ , respectively. The condition that the neighbouring solutions  $\mathbf{c} = (c_0, c_1, c_2, c_3)$  and  $\mathbf{c} + d\mathbf{c} = (c_0 + dc_0, c_1 + dc_1, c_2 + dc_2, c_3 + dc_3)$  of  $y^{(4)} = 0$  have first order contact at  $(x_0, y(x_0))$  is equivalent to the requirement that the system

$$(1.13) \quad \begin{aligned} dc_0 + 3x_0dc_1 + 3x_0^2dc_2 + x_0^3dc_3 &= 0 \\ dc_1 + 2x_0dc_2 + x_0^2dc_3 &= 0 \end{aligned}$$

has a common root  $x_0$ . Similarly, the condition that the neighbouring solutions  $\mathbf{c} = (c_0, c_1, c_2, c_3, c_4)$  and  $\mathbf{c} + d\mathbf{c} = (c_0 + dc_0, c_1 + dc_1, c_2 + dc_2, c_3 + dc_3, c_4 + dc_4)$  of  $y^{(5)} = 0$  have first order contact at  $(x_0, y(x_0))$  is equivalent to the requirement that the system

$$(1.14) \quad \begin{aligned} dc_0 + 4x_0dc_1 + 6x_0^2dc_2 + 4x_0^3dc_3 + x_0^4dc_4 &= 0 \\ dc_1 + 3x_0dc_2 + 3x_0^2dc_3 + x_0^3dc_4 &= 0 \end{aligned}$$

has a common root  $x_0$ . Calculating the resultants for the systems (1.12), (1.13), and (1.14) we get:

$$\begin{aligned} \bullet R_3 &= g(d\mathbf{c}, d\mathbf{c})dc_2 && \text{if } n = 3, \\ \bullet R_4 &= {}^4I(d\mathbf{c}, d\mathbf{c}, d\mathbf{c}, d\mathbf{c})dc_3 && \text{if } n = 4, \\ \bullet R_5 &= {}^5I(d\mathbf{c}, d\mathbf{c}, d\mathbf{c}, d\mathbf{c}, d\mathbf{c})dc_4 && \text{if } n = 5, \end{aligned}$$

where  $g$ ,  ${}^4I$  and  ${}^5I$  are as in (1.10).

This confirms our earlier statement that two neighbouring solutions of  $y''' = 0$  have first order contact iff  $g(d\mathbf{c}, d\mathbf{c}) = 0$ , since if  $dc_2 = 0$  the system (1.12) collapses to  $dc_1 = dc_0 = 0$ . Similarly, one can prove that two neighbouring solutions of  $y^{(4)} = 0$  or  $y^{(5)} = 0$  have first order contact if and only if they are *null separated* in the respective symmetric multilinear forms  ${}^4I$  or  ${}^5I$ . Our previous discussion of the invariant properties of these forms, shows that in the solution space of an ODE  $y^{(n)} = 0$ , for  $n \geq 4$ , there is a naturally defined action of the  $\mathbf{GL}(2, \mathbb{R})$  group.

This group is the stabilizer of the invariant polynomial  $I(\text{dc})$  which distinguishes neighbouring solutions having first order contact.

The main question one can ask in this context is if one can retain this  $\mathbf{GL}(2, \mathbb{R})$  structure in the solution space for more complicated ODEs. In other words, one may ask the following: What does one have to assume about the function  $F$ , defining an ODE

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}),$$

in order *to have* a well defined *conformal* tensor  $g$ ,  ${}^4I$  or  ${}^5I$ , in the respective cases  $n = 3, 4, 5$ , on the solution space of the ODE? The same question can be repeated for any  $n > 5$  and the invariant  $I$ .

The answer to this question in the  $n = 4$  case was found by Robert Bryant in [2]. Later, the answer for  $n > 4$  case was given by Boris Doubrov [7] who established a connection between the *Wilczynski invariants* [20] for a linear ODE, and certain *contact invariant conditions* for a *nonlinear* ODE associated with it. For any  $n \geq 3$ , given  $F$ , Doubrov conditions are built from the Wilczynski invariants calculated for the linearization of  $y^{(n)} = F$  about one of its solutions (see [7] for details). In a quite different perspective, these conditions, were also discovered by Maciej Dunajski and Paul K Tod [8].

Doubrov-Wilczynski conditions differ from Bryant ones for  $n = 4$ . They also differ from the conditions we are going to discuss in the present paper for  $n \geq 4$ . Doubrov, Dunajski and Tod have  $(n - 2)$  nonlinear PDEs for  $F$  of ODE  $y^{(n)} = F$ . Although this number,  $(n - 2)$ , agrees with the number of conditions we present here, there is an important difference: each of the  $(n - 2)$  conditions for  $F$ , defined by the above authors, has a *different* order in the derivatives of  $F$ . When we arrange Doubrov-Wilczynski conditions according to the order of the corresponding PDEs for  $F$ , we find that the first condition is of order 3, the second is of order 4, and so on, up to the order  $n$  of the  $(n - 2)$ -th condition. On the contrary *each of our*  $(n - 2)$  conditions is of the *third* order in the derivatives of  $F$ . The simple explanation of this discrepancy is as follows: We obtain our conditions, by applying a variant of *Cartan's equivalence method*; in the process of extracting them we obtain the first condition to be of the third order as everybody does. But the second condition which, if we were not applying Cartan's method, would be of order four, actually collapses in our derivation to a condition of order *three*. This is because Cartan's method automatically utilises the first condition of order three by differentiating it, and then eliminating the fourth derivative from the second condition by means of the fourth derivative from the differentiated condition of order three. The same situation is automatically accomplished for the condition of order *five* and so on. As a result we have  $(n - 2)$  conditions of order *three*. They are different from the Doubrov-Wilczynski conditions already for the ODE of order *four*. In the  $n = 3$  case all the conditions, namely those of Wünschmann, Doubrov, Dunajski and Tod, and ours are the same. In dimension  $n = 4$  our conditions agree with the Bryant ones. Since Wünschmann was the first who obtained these type of conditions in  $n = 3$  and found method of their constructing for arbitrary  $n$  we call the conditions discussed in this paper *generalized Wünschmann's conditions*, or *Wünschmann's conditions*, for short.

Finding the Wünschmann conditions for  $F$  of order  $n \geq 4$ , although important, is only a byproduct of our analysis. The present paper is devoted to a thorough study of the *irreducible*  $\mathbf{GL}(2, \mathbb{R})$  *geometry in dimension five*. This is done from

two points of view: first as a study of an *abstract* geometry on a manifold and, second, as a study of a *contact geometry of fifth order ODEs*.

We define an abstract 5-dimensional  $\mathbf{GL}(2, \mathbb{R})$  geometry in two steps.

First, in section 2, we show how to construct the algebraic model for the  $\mathbf{GL}(2, \mathbb{R})$  geometry in dimension five utilising properties of a *rational normal curve*. Second, instead of obtaining the reduction from  $\mathbf{GL}(5, \mathbb{R})$  to  $\mathbf{GL}(2, \mathbb{R})$  by stabilizing the 6-tensor  ${}^5I$ , we get the desired reduction by stabilizing a conformal metric  $g_{ij} \rightarrow e^{2\phi} g_{ij}$  of signature  $(3, 2)$  and a conformal totally symmetric 3-tensor  $\Upsilon_{ijk} \rightarrow e^{3\phi} \Upsilon_{ijk}$ . These tensors are supposed to be related by the following algebraic constraint:

$$(1.15) \quad g^{lm}(\Upsilon_{ijl}\Upsilon_{kmp} + \Upsilon_{kil}\Upsilon_{jmp} + \Upsilon_{jkl}\Upsilon_{imp}) = g_{ij}g_{kp} + g_{kl}g_{jp} + g_{jk}g_{ip}.$$

It is worthwhile to note that condition (1.15) is a non-Riemannian counterpart of the condition considered by Elie Cartan in the context of isoparametric surfaces [3], [4].

Our main object of study is then defined as follows:

*Definition.* An irreducible  $\mathbf{GL}(2, \mathbb{R})$  geometry in dimension five is a 5-dimensional manifold  $M^5$  equipped with a class of triples  $[g, \Upsilon, A]$  such that on  $M^5$ :

- (a)  $g$  is a metric of signature  $(3, 2)$ ,
- (b)  $\Upsilon$  is a traceless symmetric 3rd rank tensor,
- (c)  $A$  is a 1-form,
- (d) the metric  $g$  and the tensor  $\Upsilon$  satisfy the identity (1.15),
- (e) two triples  $(g, \Upsilon, A)$  and  $(g', \Upsilon', A')$  are in the same class  $[g, \Upsilon, A]$  if and only if there exists a function  $\phi : M^5 \rightarrow \mathbb{R}$  such that

$$g' = e^{2\phi} g, \quad \Upsilon' = e^{3\phi} \Upsilon, \quad A' = A - 2d\phi.$$

This definition places  $\mathbf{GL}(2, \mathbb{R})$  geometries in dimension five among the *Weyl geometries*  $[g, A]$ . They are special Weyl geometries i.e. such for which the structure group is reduced from  $\mathbf{CO}(3, 2)$  to  $\mathbf{GL}(2, \mathbb{R})$ . A natural description of such geometries should be then obtained in terms of a certain  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection. However, unlike the usual Weyl case, the choice of such a connection is ambiguous, due to the fact that such a connection has non-vanishing torsion in general, and one must find admissible conditions for the torsion that specify connection uniquely. Pursuing this problem in section 3 we find an interesting subclass of  $\mathbf{GL}(2, \mathbb{R})$  geometries.

*Proposition.* A  $\mathbf{GL}(2, \mathbb{R})$  geometry  $[g, \Upsilon, A]$  is called nearly integrable if the Weyl connection  $\overset{W}{\nabla}$  of  $[g, A]$  satisfies

$$(\overset{W}{\nabla}_X \Upsilon)(X, X, X) = -\frac{1}{2}A(X)\Upsilon(X, X, X).$$

It turns out, see section 3, that the nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  geometries *uniquely* define a  $\mathfrak{gl}(2, \mathbb{R})$  connection  $D$ . This is characterized by the following requirements:

- it preserves the structural tensors:

$$\begin{aligned} Dg_{ij} &= -Ag_{ij}, \\ D\Upsilon_{ijk} &= -\frac{3}{2}A\Upsilon_{ijk}, \end{aligned}$$

- and it has *totally skew symmetric torsion*.



We call this unique connection the *characteristic* connection for the nearly integrable structure  $\mathbf{GL}(2, \mathbb{R})$ .

The rest of section 3 is devoted to study the algebraic structure of the torsion and the curvature of the characteristic connection of a nearly integrable structure. Since the tensor products of tangent spaces are reducible under the action of  $\mathbf{GL}(2, \mathbb{R})$ , we decompose the torsion and the curvature tensors into components belonging to the irreducible representations. In particular, the skew symmetric torsion  $T$  has two components,  $T^{(3)}$  and  $T^{(7)}$ , lying in the three-dimensional and the seven-dimensional irreducible representations respectively. Likewise the Maxwell 2-form  $dA$  and the Ricci tensor  $R$  decompose according to  $dA = dA^{(3)} + dA^{(7)}$  and  $R = R^{(1)} + R^{(3)} + R^{(5)} + R^{(7)} + R^{(9)}$ . The last problem we address in section 3 concerns with the properties of geometries whose characteristic connections have ‘the smallest possible’ torsion, that is the torsion of the pure three-dimensional type. In proposition 3.11 we prove that Ricci tensor for such structures satisfies remarkable identities:

$$R^{(3)} = \frac{1}{4}dA^{(3)}, \quad R^{(7)} = \frac{3}{2}dA^{(7)}, \quad R^{(9)} = 0.$$

The third equation is equivalent to

$$R_{(ij)} = \frac{1}{5}Rg_{ij} + \frac{2}{7}R_{kl}\Upsilon^{klm}\Upsilon_{ijm}.$$

In section 4 we briefly describe  $\mathbf{GL}(2, \mathbb{R})$  geometry in the language of the bundle  $\mathbf{GL}(2, \mathbb{R}) \rightarrow P \rightarrow M^5$ . We also show how an appropriate coframe defined on a nine-dimensional manifold  $P$  turns this manifold into a bundle  $\mathbf{GL}(2, \mathbb{R}) \rightarrow P \rightarrow M^5$  and generates the  $\mathbf{GL}(2, \mathbb{R})$  geometry on  $M^5$ . This construction is the core of the proof of the main theorem in section 5. This closes the part of the paper that is devoted to abstract  $\mathbf{GL}(2, \mathbb{R})$  geometries.

Section 5 contains the main result of this paper, theorem 5.3, which links  $\mathbf{GL}(2, \mathbb{R})$  geometry with the realm of ordinary differential equations. It can be encapsulated as follows.

*Theorem.* A 5th order ODE  $y^{(5)} = F(x, y, y', y'', y''', y^{(4)})$  that satisfies three Wünschmann conditions defines a nearly integrable irreducible  $\mathbf{GL}(2, \mathbb{R})$  geometry  $(M^5, [g, \Upsilon, A])$  on the space  $M^5$  of its solutions. This geometry has the characteristic connection with torsion of the ‘pure’ type in the 3-dimensional irreducible representation of  $\mathbf{GL}(2, \mathbb{R})$ . Two 5th order ODEs that are equivalent under contact transformation of variables define equivalent  $\mathbf{GL}(2, \mathbb{R})$  geometries.

The theorem has numerous applications. For example, we use it to characterise various classes of Wünschmann 5th order ODEs, by means of the algebraic type of the tensors associated with the corresponding characteristic connection. For example iff  $F_{y^{(4)}y^{(4)}} = 0$ , the torsion of the characteristic connection vanishes, and iff  $F_{y^{(4)}y^{(4)}y^{(4)}} = 0$ , then we have  $dA^{(7)} = 0$ .

The proof of the theorem consists of an application of the Cartan method of equivalence. We write an ODE, considered modulo contact transformation of variables, as a  $G$ -structure on the four-order jet space. Starting from this  $G$ -structure we explicitly construct a 9-dimensional manifold  $P$ , which is a  $\mathbf{GL}(2, \mathbb{R})$  bundle over the solution space and carries certain distinguished coframe. This construction is only possible provided that the ODE satisfies the Wünschmann conditions, which we write down explicitly. By means of proposition 4.1 the coframe on  $P$

defines the nearly integrable geometry on the solution space of the ODE. It has the characteristic connection with torsion in the 3-dimensional representation.

Section 6 includes examples of 5th order equations in the Wünschmann class. We find equations generating all the structures with vanishing torsion, equations possessing at least 6-dimensional group of contact symmetries and yielding geometries with  $dA = 0$ . We also give highly nontrivial examples of equations for which  $dA \neq 0$ , including a family of examples with function  $F$  being a solution of a certain second order ODE.

Finally, in section 7 we consider ODEs of order  $n > 5$ . We apply results of the Hilbert theory of algebraic invariants, to define the tensors responsible for the reduction  $\mathbf{GL}(n, \mathbb{R}) \rightarrow \mathbf{GL}(2, \mathbb{R})$ . We also give the explicit formulae for the  $(n-2)$  third order Wünschmann conditions for  $n = 6$  and  $n = 7$ .

## 2. A PECULIAR THIRD RANK SYMMETRIC TENSOR

Consider  $\mathbb{R}^n$  equipped with a *Riemannian* metric  $g$  and a 3rd rank tracefree symmetric tensor  $\Upsilon \in S_0^3 \mathbb{R}^n$  satisfying:

- (i)  $\Upsilon_{ijk} = \Upsilon_{(ijk)}$  - (symmetry)
- (ii)  $g^{ij} \Upsilon_{ijk} = 0$  - (tracefree)
- (iii)  $g^{lm} (\Upsilon_{ijl} \Upsilon_{kmp} + \Upsilon_{kil} \Upsilon_{jmp} + \Upsilon_{jkl} \Upsilon_{imp}) = g_{ij} g_{kp} + g_{kl} g_{jp} + g_{jk} g_{ip}$ .

It turns out that the third condition is very restrictive. In particular Cartan has shown [3, 4] that for (iii) to be satisfied the dimension  $n$  must be one of the following:  $n = 5, 8, 14, 26$ . Moreover Cartan constructed  $\Upsilon$  in each of these dimensions and has shown that it is unique up to an  $\mathbf{O}(n)$  transformation. Restricting to  $n = 5, 8, 14, 26$ , we consider the stabilizer  $H_n$  of  $\Upsilon$  under the action of  $\mathbf{GL}(n, \mathbb{R})$ :

$$H_n = \{\mathbf{GL}(n, \mathbb{R}) \ni a : \Upsilon(aX, aY, aZ) = \Upsilon(X, Y, Z), \forall X, Y, Z \in \mathbb{R}^n\}.$$

Then, one finds that:

- $H_5 = \mathbf{SO}(3) \subset \mathbf{SO}(5)$  in the 5-dimensional irreducible representation,
- $H_8 = \mathbf{SU}(3) \subset \mathbf{SO}(8)$  in the 8-dimensional irreducible representation,
- $H_{14} = \mathbf{Sp}(3) \subset \mathbf{SO}(14)$  in the 14-dimensional irreducible representation,
- $H_{26} = \mathbf{F}_4 \subset \mathbf{SO}(26)$  in the 26-dimensional irreducible representation.

The relevance of conditions (i)-(iii) is that they are invariant under the  $\mathbf{O}(n)$  action on the space of tracefree symmetric tensors  $S_0^3 \mathbb{R}^n$ . Moreover they totally characterize the orbit  $\mathbf{O}(n)/H_n \subset S_0^3$  of the tensor  $\Upsilon$  under this action [1, 16].

The question arises if one can construct tensors satisfying (i)-(iii) for metrics having non-Riemannian signatures. Below we show how to do it if  $n = 5$  and the metric  $g$  has signature  $(3, 2)$ . This construction described to us by E. Ferapontov [10, 11] is as follows.

Consider  $\mathbb{R}^5$  with coordinates  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$ , and a curve

$$\gamma(x) = (1, x, x^2, x^3, x^4) \subset \mathbb{R}^5.$$

Associated to the curve  $\gamma$  there are *two* algebraic varieties in  $\mathbb{R}^5$ :

- *The bisecant variety.* This is defined to be a set consisting of all the points on all straight lines *crossing the curve*  $\gamma$  in precisely *two* points. It is given parametrically as

$$B(x, s, u) = (1, x, x^2, x^3, x^4) + u(0, x - s, x^2 - s^2, x^3 - s^3, x^4 - s^4),$$

where  $x, s, u$  are *three* real parameters.

- *The tangent variety.* This is defined to be a set consisting of all the points on all straight lines *tangent* to the curve  $\gamma$ . It is given parametrically as

$$T(x, s) = (1, x, x^2, x^3, x^4) + s(0, 1, 2x, 3x^2, 4x^3).$$

One of many interesting features of these two varieties is that they define (up to a scale) a tri-linear symmetric form

$$(2.1) \quad \Upsilon(\theta) = 3\sqrt{3}(\theta^0\theta^2\theta^4 + 2\theta^1\theta^2\theta^3 - (\theta^2)^3 - \theta^0(\theta^3)^2 - \theta^4(\theta^1)^2)$$

and a bi-linear symmetric form

$$(2.2) \quad g(\theta) = \theta^0\theta^4 - 4\theta^1\theta^3 + 3(\theta^2)^2.$$

These forms are distinguished by the fact that the bisecant and tangent varieties are contained in their null cones. In the homogeneous coordinates  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$  in  $\mathbb{R}^5$  all the points  $\theta$  of  $B(x, s, u)$  satisfy

$$\Upsilon(\theta) = 0,$$

whereas all the points  $\theta$  of  $T(x, s)$  satisfy

$$\Upsilon(\theta) = 0 \quad \text{and} \quad g(\theta) = 0.$$

Writing the forms as  $\Upsilon(\theta) = \Upsilon_{ijk}\theta^i\theta^j\theta^k$ ,  $g(\theta) = g_{ij}\theta^i\theta^j$ ,  $i, j, k = 0, 1, 2, 3, 4$  one can check that so defined  $g_{ij}$  and  $\Upsilon_{ijk}$  satisfy relations (i)-(iii) of the previous section.

Although it is obvious we remark that the above defined metric  $g_{ij}$  has signature  $(3, 2)$ .

As we have already noted the forms  $\Upsilon(\theta)$  and  $g(\theta)$  are defined only *up to a scale*. We were also able to find a factor, the  $3\sqrt{3}$  in expression (2.1), that makes the corresponding  $g_{ij}$  and  $\Upsilon_{ijk}$  satisfy (i)-(iii). Note that these conditions are *conformal* under the simultaneous change:

$$g_{ij} \rightarrow e^{2\phi}g_{ij}, \quad \Upsilon_{ijk} \rightarrow e^{3\phi}\Upsilon_{ijk}.$$

Thus it is interesting to consider in  $\mathbb{R}^5$  a *class* of pairs  $[g, \Upsilon]$ , such that:

- in each pair  $(g, \Upsilon)$ 
  - $g$  is a metric of signature  $(3, 2)$ ,
  - $\Upsilon$  is a traceless symmetric 3rd rank tensor,
  - the metric  $g$  and the tensor  $\Upsilon$  satisfy the identity

$$g^{lm}(\Upsilon_{ijl}\Upsilon_{kmp} + \Upsilon_{kil}\Upsilon_{jmp} + \Upsilon_{jkl}\Upsilon_{imp}) = g_{ij}g_{kp} + g_{kl}g_{jp} + g_{jk}g_{ip},$$

- two pairs  $(g, \Upsilon)$  and  $(g', \Upsilon')$  are in the same class  $[g, \Upsilon]$  if and only if there exists  $\phi \in \mathbb{R}$  such that

$$(2.3) \quad g' = e^{2\phi}g, \quad \Upsilon' = e^{3\phi}\Upsilon.$$

Given a structure  $(\mathbb{R}^5, [g, \Upsilon])$  we define a group  $CH$  to be a subgroup of the general linear group  $\mathbf{GL}(5, \mathbb{R})$  preserving  $[\Upsilon]$ . This means that, choosing a representative  $\Upsilon$  of the class  $[\Upsilon]$ , we define  $CH$  to be:

$$CH = \{\mathbf{GL}(5, \mathbb{R}) \ni a : \Upsilon(ax, ax, ax) = (\det a)^{(3/5)}\Upsilon(x, x, x)\}.$$

Note that the exponent  $\frac{3}{5}$  in the above expression is caused by the fact that the r.h.s. of the equation defining the group elements must be *homogeneous of degree 3* in  $a$ , similarly as the l.h.s. is.

This definition does not depend on the choice of a representative  $\Upsilon \in [\Upsilon]$ . We have the following

**Proposition 2.1.** *The set  $CH$  of  $5 \times 5$  real matrices  $a \in \mathbf{GL}(5, \mathbb{R})$  preserving  $[\Upsilon]$  is the  $\mathbf{GL}(2, \mathbb{R})$  group in its 5-dimensional irreducible representation. Moreover, we have natural inclusions*

$$CH = \mathbf{GL}(2, \mathbb{R}) \subset \mathbf{CO}(3, 2) \subset \mathbf{GL}(5, \mathbb{R}),$$

where  $\mathbf{CO}(3, 2)$  is the 11-dimensional group of homotheties associated with the conformal class  $[g]$ .

*Remark 2.2.* According to our Introduction, there is another  $\mathbf{GL}(2, \mathbb{R})$  invariant symmetric conformal tensor that stabilizes  $\mathbf{GL}(5, \mathbb{R})$  to the irreducible  $\mathbf{GL}(2, \mathbb{R})$ . This is the tensor  ${}^5I_{ijklpq}$  defined via  ${}^5I(\theta) = \frac{1}{120} {}^5I_{ijklpq} \theta^i \theta^j \theta^k \theta^l \theta^p \theta^q$  with  ${}^5I$  as in (1.10). We prefer however to work with a pair  $(g_{ij}, \Upsilon_{ijk})$  rather than with  ${}^5I_{ijklpq}$ , because of the lower rank, and more importantly, because of the evident conformal *metric* properties of the  $(g_{ij}, \Upsilon_{ijk})$  approach. Also, it is worthwhile to note that the invariants  $g_{ij}$ ,  $\Upsilon_{ijk}$  and  ${}^5I_{ijklpq}$  are *not* independent. Indeed, one can easily check that  ${}^5I$  of (1.10),  $\Upsilon$  of (2.1) and  $g$  of (2.2) are related by  ${}^5I = \Upsilon^2 - g^3$ . We interpret this relation as the definition of  ${}^5I$  in terms of more primitive quantities  $g$  and  $\Upsilon$ .

The isotropy condition for the group elements  $a$  of  $CH$  has its obvious counterpart at the level of the Lie algebra  $\mathfrak{gl}(2, \mathbb{R}) = (\mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})) \subset \mathfrak{co}(3, 2) \subset \mathfrak{gl}(5, \mathbb{R})$  of  $CH = \mathbf{GL}(2, \mathbb{R})$ . Writing  $a = \exp(t\Gamma)$  we find that the infinitesimal version of the isotropy condition, written in terms of the  $5 \times 5$  matrices  $\Gamma = (\Gamma^i_j)$  is:

$$(2.4) \quad \Gamma^l_i \Upsilon_{ljk} + \Gamma^l_j \Upsilon_{ilk} + \Gamma^l_k \Upsilon_{ijl} = \frac{3}{5} \text{Tr}(\Gamma) \Upsilon_{ijk},$$

where  $\text{Tr}(\Gamma) = \Gamma^m_m$ . Given  $\Upsilon_{ijk}$ , these *linear* equations can be solved for  $\Gamma$ . Taking the most general matrix  $\Gamma \in \mathbf{GL}(5, \mathbb{R})$  and  $\Upsilon_{ijk}$  given by  $\Upsilon(x, x, x) = \Upsilon_{ijk} x^i x^j x^k$  of (2.1) we find the explicit realization of the 5-dimensional representation of the  $\mathfrak{gl}(2, \mathbb{R})$  Lie algebra as:

$$(2.5) \quad \Gamma = \Gamma_- E_- + \Gamma_+ E_+ + \Gamma_0 E_0 + \Gamma_1 E_1,$$

where  $\Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1$  are free real parameters, and  $(E_-, E_+, E_0, E_1)$  are  $5 \times 5$  matrices given by:

$$(2.6) \quad E_+ = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix},$$

$$E_0 = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \quad E_1 = -4 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The commutator in

$$\mathfrak{gl}(2, \mathbb{R}) = \text{Span}_{\mathbb{R}}(E_-, E_+, E_0, E_1)$$

is the usual commutator of matrices. In particular, the non-vanishing commutators are:

$$[E_0, E_+] = -2E_+ \quad , \quad [E_0, E_-] = 2E_- \quad , \quad [E_+, E_-] = -E_0.$$

Note that

$$\mathfrak{sl}(2, \mathbb{R}) = \text{Span}_{\mathbb{R}}(E_-, E_+, E_0)$$

is a subalgebra of  $\mathfrak{gl}(2, \mathbb{R})$  isomorphic to  $\mathfrak{sl}(2, \mathbb{R})$ . It provides the 5-dimensional irreducible representation of  $\mathfrak{sl}(2, \mathbb{R})$ .

### 3. IRREDUCIBLE $\mathbf{GL}(2, \mathbb{R})$ GEOMETRIES IN DIMENSION FIVE

In this section we describe 5-dimensional manifolds whose tangent space at each point is equipped with the structure  $[g, \Upsilon]$  of the previous section. We will analyze such manifolds in terms of an appropriately chosen connection. We will describe connections on a manifold  $M$  in terms of Lie-algebra-valued 1-forms on  $M$ . To be more specific, let  $\dim M = n$  and let  $\mathfrak{g}$  denote an  $n$ -dimensional representation of some Lie algebra. The connection 1-forms  $\Gamma^i_j$  on  $M$  are the matrix entries of an element  $\Gamma \in \mathfrak{g} \otimes \Lambda^1 M$ . They define the covariant exterior derivative  $D$ . This acts on tensor-valued-forms via the extension to the higher order tensors of the formula:

$$Dv^i = dv^i + \Gamma^i_j \wedge v^j.$$

Now suppose that we have a 5-dimensional manifold  $M^5$  equipped with a class of pairs  $[g, \Upsilon]$  such that  $g$  is a metric,  $\Upsilon$  is a 3rd rank tensor related to the metric via properties (i)-(iii) of the previous section, and two pairs  $(g, \Upsilon)$  and  $(g', \Upsilon')$  are in the same pair iff they are related by (2.3), where  $\phi$  is now a *function* on  $M^5$ . If we want to associate a connection with such a structure we have to specify how this connection is related to the pair  $[g, \Upsilon]$ . A possible approach is to choose a representative  $(g, \Upsilon)$  of  $[g, \Upsilon]$  and declare what is  $Dg$  and  $D\Upsilon$ . A first possible choice  $Dg = 0$  or  $D\Upsilon = 0$  is definitely not good since, in general,  $Dg'$  and  $D\Upsilon'$  would not be vanishing for another choice of the representative of  $[g, \Upsilon]$ . A remedy for this situation comes from the *Weyl geometry* where, given a conformal class  $(M, [g])$ , a 1-form  $A$  is introduced so that the connection satisfies  $Dg_{ij} = -Ag_{ij}$ . In our case we introduce a 1-form  $A$  on  $M^5$  and require that  $Dg_{ij} = -Ag_{ij}$  and  $D\Upsilon_{ijk} = -\frac{3}{2}A\Upsilon_{ijk}$ . Then, if we transform  $(g, \Upsilon)$  according to (2.3), the transformed objects will satisfy  $Dg'_{ij} = -A'g'_{ij}$  and  $D\Upsilon'_{ijk} = -\frac{3}{2}A'\Upsilon'_{ijk}$  provided that  $A' = A - 2d\phi$ . This motivates the following

**Definition 3.1.** An irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure in dimension five is a 5-dimensional manifold  $M^5$  equipped with a class of triples  $[g, \Upsilon, A]$  such that on  $M^5$ :

- (a)  $g$  is a metric of signature  $(3, 2)$ ,
- (b)  $\Upsilon$  is a traceless symmetric 3rd rank tensor,
- (c)  $A$  is a 1-form,
- (d) the metric  $g$  and the tensor  $\Upsilon$  satisfy the identity

$$g^{lm}(\Upsilon_{ijl}\Upsilon_{kmp} + \Upsilon_{kil}\Upsilon_{jmp} + \Upsilon_{jkl}\Upsilon_{imp}) = g_{ij}g_{kp} + g_{kl}g_{jp} + g_{jk}g_{ip},$$

- (e) two triples  $(g, \Upsilon, A)$  and  $(g', \Upsilon', A')$  are in the same class  $[g, \Upsilon, A]$  if and only if there exists a function  $\phi : M^5 \rightarrow \mathbb{R}$  such that

$$g' = e^{2\phi}g, \quad \Upsilon' = e^{3\phi}\Upsilon, \quad A' = A - 2d\phi.$$

If  $M^5$  was only equipped with a class of *pairs*  $[g, A]$  satisfying conditions (a), (c) and (e) (with  $\Upsilon, \Upsilon'$  omitted), then  $(M^5, [g, A])$  would define the *Weyl* geometry. Such geometry, which has the structure group  $\mathbf{CO}(3, 2)$ , is usually studied in terms of the *Weyl connection*. This is the *unique* torsionfree connection preserving the conformal structure  $[g, A]$ . It is defined by the following two equations:

$$(3.1) \quad \overset{W}{D} g_{ij} = -A g_{ij} \quad (\text{preservation of the class } [g, A]),$$

$$(3.2) \quad \overset{W}{D} \theta^i = 0 \quad (\text{no torsion}),$$

where  $\theta^i$  is a coframe related to the representative  $g$  of the class  $[g]$  by  $g = g_{ij} \theta^i \theta^j$ . We describe the Weyl connection in terms of the Weyl connection 1-forms  $\overset{W}{\Gamma}^i_j$ ,  $i, j = 0, 1, 2, 3, 4$ .

Take a representative  $(g, A)$  of the Weyl structure  $[g, A]$  on  $M^5$ . Choose a coframe  $(\theta^i)$ ,  $i = 0, 1, 2, 3, 4$ , such that  $g = g_{ij} \theta^i \theta^j$ , with all the metric coefficients  $g_{ij}$  being *constant*. Then the above two equations define  $\overset{W}{\Gamma}^i_j$  together with  $\overset{W}{\Gamma}_{ij} = g_{ik} \overset{W}{\Gamma}^k_j$  to be 1-forms on  $M^5$  satisfying

$$(3.3) \quad \overset{W}{\Gamma}_{ij} + \overset{W}{\Gamma}_{ji} = A g_{ij} \quad (\text{preservation of the class } [g, A]),$$

$$(3.4) \quad d\theta^i + \overset{W}{\Gamma}^i_j \wedge \theta^j = 0 \quad (\text{no torsion}).$$

It follows that once the representative  $(g, A)$  and the coframe  $\theta^i$  is chosen the above equations *uniquely* determine the Weyl connection 1-forms  $\overset{W}{\Gamma}^i_j$ .

We note that, due to condition (3.3), matrix  $\overset{W}{\Gamma} = (\overset{W}{\Gamma}^i_j)$  of the Weyl connection 1-forms belongs to the 5-dimensional defining representation of the Lie algebra  $\mathfrak{co}(3, 2) \subset \text{End}(5, \mathbb{R})$  of the Lie group  $\mathbf{CO}(3, 2) \subset \mathbf{GL}(5, \mathbb{R})$ . Consequently, the Weyl connection coefficients  $\overset{W}{\Gamma}^i_{jk}$ , defined by  $\overset{W}{\Gamma}^i_j = \overset{W}{\Gamma}^i_{jk} \theta^k$  belong to the tensor product  $\mathfrak{co}(3, 2) \otimes \mathbb{R}^5$ , the vector space of dimension  $(1+10)5=55$ .

Now we assume that we have an irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure  $[g, \Upsilon, A]$  on a 5-manifold  $M^5$ . Forgetting about  $\Upsilon$  gives the Weyl geometry as before. In particular there is the *unique* Weyl connection  $\overset{W}{\Gamma}$  associated with  $[g, \Upsilon, A]$ . But the existence of a metric compatible class of tensors  $\Upsilon$  makes this Weyl geometry more special. To analyze it we introduce a *new* connection, which will be respecting the entire structure  $[g, \Upsilon, A]$ . This is rather a complicated procedure which we describe below.

Firstly we require that the new connection preserves  $[g]$  and  $[\Upsilon]$ :

$$(3.5) \quad Dg_{ij} = -A g_{ij}$$

$$(3.6) \quad D\Upsilon_{ijk} = -\frac{3}{2} A \Upsilon_{ijk}.$$

This does not determine the connection uniquely – to have the uniqueness we need to specify what the torsion of  $D$  is. We need some preparations to discuss it.

**Definition 3.2.** Let  $(g, \Upsilon, A)$  be a representative of an irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure on a 5-dimensional manifold  $M^5$ . A coframe  $\theta^i$ ,  $i = 0, 1, 2, 3, 4$ , on  $M^5$  is called *adapted* to the representative  $(g, \Upsilon, A)$  if

$$g = g_{ij} \theta^i \theta^j = \theta^0 \theta^4 - 4\theta^1 \theta^3 + 3(\theta^2)^2$$

and

$$\Upsilon = \Upsilon_{ijk} \theta^i \theta^j \theta^k = 3\sqrt{3}(\theta^0 \theta^2 \theta^4 + 2\theta^1 \theta^2 \theta^3 - (\theta^2)^3 - \theta^0 (\theta^3)^2 - \theta^4 (\theta^1)^2).$$

Locally such a coframe always exists and is given up to a  $\mathbf{GL}(2, \mathbb{R})$  transformation.

Let us now choose an adapted coframe  $\theta^i$  to a representative  $(g, \Upsilon, A)$  of  $[g, \Upsilon, A]$ . In this coframe equations (3.5)-(3.6) can be rewritten in terms of the connection 1-forms  $\Gamma^i_j$  as

$$(3.7) \quad \Gamma^l_i g_{lj} + \Gamma^l_j g_{li} = A g_{ij}$$

$$(3.8) \quad \Gamma^l_i \Upsilon_{ljk} + \Gamma^l_j \Upsilon_{ilk} + \Gamma^l_k \Upsilon_{ijl} = \frac{3}{2} A \Upsilon_{ijk}.$$

When we contract the first equation in indices  $i$  and  $j$  we get

$$(3.9) \quad A = \frac{2}{5} \Gamma^l_l = \frac{2}{5} \text{Tr}(\Gamma).$$

Inserting this into (3.8) we get

$$(3.10) \quad \Gamma^l_i \Upsilon_{ljk} + \Gamma^l_j \Upsilon_{ilk} + \Gamma^l_k \Upsilon_{ijl} = \frac{3}{5} \Gamma^l_l \Upsilon_{ijk}.$$

Comparing this with (2.4) we see that the *general solution* for the connection 1-forms  $\Gamma^i_j$  are given by (2.5), i.e.

$$\Gamma = \Gamma_- E_- + \Gamma_+ E_+ + \Gamma_0 E_0 + \Gamma_1 E_1,$$

where  $(\Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$  are four 1-forms on  $M^5$  such that

$$(3.11) \quad \Gamma_1 = -\frac{1}{8} A.$$

To fix the remaining three 1-forms  $(\Gamma_-, \Gamma_+, \Gamma_1)$  we introduce an operator

$$\tilde{\Upsilon} : \mathfrak{co}(3, 2) \otimes \mathbb{R}^5 \rightarrow S^4 \mathbb{R}^5$$

defined by:

$$\tilde{\Upsilon}(\Gamma)_{ijkm}^W = \Upsilon_{l(ij} \Gamma_{km)}^l - \frac{1}{5} \Gamma_{l(m}^l \Upsilon_{ijk)},$$

and analyze its kernel  $\ker \tilde{\Upsilon}$ .

Writing equation (3.10) in terms of the coefficients  $\Gamma_{im}^l \in \mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5$  and symmetrizing it over the indices  $\{imjk\}$ , we see that the *whole*  $\mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5$  is included in  $\ker \tilde{\Upsilon}$ .

We use the metric to identify  $\mathbb{R}^5$  with  $(\mathbb{R}^5)^*$ , and more generally to identify tensor spaces  $\bigotimes^k (\mathbb{R}^5)^* \bigotimes^l \mathbb{R}^5$  with  $\bigotimes^{(k+l)} (\mathbb{R}^5)^*$ . This enables us to identify the objects with upper indices with the corresponding objects with lower indices, e.g.  $T_{ijk} = g_{il} T_{jk}^l$ . Having in mind these identifications we easily see that, due to the antisymmetry in the last two indices, every 3-form  $T_{ijk} = T_{[ijk]}$  is included in  $\ker \tilde{\Upsilon}$ .

Thus we have:

$$\mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5 \subset \ker \tilde{\Upsilon},$$

$$\bigwedge^3 \mathbb{R}^5 \subset \ker \tilde{\Upsilon}.$$

The following proposition can be checked by a direct calculation involving the explicit form of the  $\mathfrak{gl}(2, \mathbb{R})$  representation given in (2.5), (2.6).

**Proposition 3.3.** *The vector space  $\ker \tilde{\Upsilon}$  has the following properties:*

$$\ker \tilde{\Upsilon} = (\mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5) \oplus \bigwedge^3 \mathbb{R}^5$$

and

$$\dim \ker \tilde{\Upsilon} = 30.$$

Now we interpret the condition  $\overset{W}{\Gamma}_{im}^l \in \ker \bar{\Upsilon}$ , i. e. the equation

$$(3.12) \quad \Upsilon_{l(ij} \overset{W}{\Gamma}_{km)}^l = \frac{1}{5} \overset{W}{\Gamma}_{l(m}^l \Upsilon_{ijk)},$$

as a *restriction* on possible Weyl connections. Let us assume that we have a structure  $(M^5, [g, \Upsilon, A])$  with the Weyl connection coefficients  $\overset{W}{\Gamma}_{im}^l$  satisfying (3.12). The coefficients  $\overset{W}{\Gamma}_{im}^l$  are written in a coframe adapted to some choice  $(g, \Upsilon, A)$ . It is easy to see, using (3.3) and contracting (3.12) over all the free indices with a vector field  $X^i$ , that the restriction on the Weyl connection (3.12) in coordinate-free language is equivalent to

$$(3.13) \quad (\overset{W}{\nabla}_X \Upsilon)(X, X, X) = -\frac{1}{2}A(X)\Upsilon(X, X, X).$$

Here  $\overset{W}{\nabla}$  denotes the Weyl connection in the Koszul notation.

**Definition 3.4.** An irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure  $(M^5, [g, \Upsilon, A])$  is called *nearly integrable* iff its Weyl connection  $\overset{W}{\nabla}$  associated to the class  $[g, A]$  satisfies (3.13).

A nice feature of nearly integrable structures  $(M^5, [g, \Upsilon, A])$  is that they define the *unique*  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection  $\Gamma$ . This follows from the above discussion about the kernel of  $\bar{\Upsilon}$ . Indeed, given a nearly integrable structure  $(M^5, [g, \Upsilon, A])$  it is enough to choose a representative  $(g, \Upsilon, A)$  and to write the equation (3.13) for the Weyl connection  $\overset{W}{\Gamma}$  in an adapted coframe  $\theta^i$ . Then the uniquely given Weyl connection coefficients  $\overset{W}{\Gamma}_{ijk}$  are by definition in  $\ker \bar{\Upsilon} = (\mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5) \oplus \bigwedge^3 \mathbb{R}^5$ , which means that they *uniquely* split onto  $\Gamma_{ijk} \in \mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5$  and  $\frac{1}{2}T_{ijk} \in \bigwedge^3 \mathbb{R}^5$ . Thus, for all nearly integrable structures  $(M^5, [g, \Upsilon, A])$ , in a coframe adapted to  $(g, \Upsilon, A)$ , we have

$$(3.14) \quad \overset{W}{\Gamma}_{ijk} = \Gamma_{ijk} + \frac{1}{2}T_{ijk},$$

and both  $\Gamma_{ijk} \in \mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5$  and  $T_{ijk} \in \bigwedge^3 \mathbb{R}^5$  are uniquely determined in terms of  $\overset{W}{\Gamma}_{ijk}$ . Now we rewrite the torsionfree condition (3.4) for the *Weyl* connection in the form

$$(3.15) \quad d\theta^i + \Gamma^i_j \wedge \theta^j = \frac{1}{2}T_{jk}^i \theta^j \wedge \theta^k.$$

It can be interpreted as follows: The nearly integrable structure  $(M^5, [g, \Upsilon, A])$ , via (3.14), uniquely determines  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection  $\Gamma_{ijk}$  which respects the structure  $[g, \Upsilon, A]$  due to (3.5), (3.6), and has *totally skew symmetric torsion*  $T_{ijk}$  due to (3.15).

We summarize this part of our considerations in the following

**Proposition 3.5.** *Every nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  structure  $(M^5, [g, \Upsilon, A])$  defines a unique  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection which has totally skew symmetric torsion.*

Also the converse is true:

**Proposition 3.6.** *Let  $(M^5, [g, \Upsilon, A])$  be an irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure and  $\overset{W}{\Gamma}_{ijk}$  be the Weyl connection coefficients associated, in an adapted coframe  $\theta^i$ , with the Weyl structure  $[g, A]$ . Assume that the Weyl structure  $[g, A]$  admits a split*

$$\overset{W}{\Gamma}_{ijk} = \Gamma_{ijk} + \frac{1}{2}T_{ijk},$$



in which  $\Gamma_{ijk} \in \mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5$  and  $T_{ijk} \in \wedge^3 \mathbb{R}^5$ . Then  $[g, \Upsilon, A]$  is nearly integrable, the split is unique and  $\Gamma_{ij} = \Gamma_{ijk} \theta^k$  is a  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection with totally skew symmetric torsion  $\Theta_i = \frac{1}{2} T_{ijk} \theta^j \wedge \theta^k$ .

**Definition 3.7.** The unique  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection with totally skew symmetric torsion naturally associated with a nearly integrable structure  $(M^5, [g, \Upsilon, A])$  is called the *characteristic connection*.

In the next paragraph we analyze algebraic structure of torsion and curvature of characteristic connections.

**3.1. Nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  structures.** Let  $(M^5, [g, \Upsilon, A])$  be a nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  structure and let  $\Gamma$  be its characteristic connection. Then the  $\mathbf{GL}(2, \mathbb{R})$  invariant information about  $(M^5, [g, \Upsilon, A])$  is encoded in its totally skew symmetric torsion  $\Theta_i = \frac{1}{2} T_{ijk} \theta^j \wedge \theta^k$  and its curvature

$$\Omega_{ij} = \frac{1}{2} R_{ijkl} \theta^k \wedge \theta^l = d\Gamma_{ij} + \Gamma_{ik} \wedge \Gamma_{kj}.$$

The spaces  $\wedge^3 \mathbb{R}^5$  and  $\mathfrak{gl}(2, \mathbb{R}) \otimes \wedge^2 \mathbb{R}^5$  are *reducible* under the action of  $\mathbf{GL}(2, \mathbb{R})$ . Their decompositions into the  $\mathbf{GL}(2, \mathbb{R})$  irreducible components may be used to classify the torsion types, in the case of  $\wedge^3 \mathbb{R}^5$ , and the curvature types, in the case of  $\mathfrak{gl}(2, \mathbb{R}) \otimes \wedge^2 \mathbb{R}^5$ . In particular, to decompose  $\wedge^3 \mathbb{R}^5$  we use the Hodge star operation associated with one of the metrics  $g$  from the class  $[g, \Upsilon, A]$ . This identifies  $\wedge^3 \mathbb{R}^5$  with  $\wedge^2 \mathbb{R}^5$ . The  $\mathbf{GL}(2, \mathbb{R})$  invariant decomposition of  $\wedge^3 \mathbb{R}^5$  is then transformed to the decomposition of  $\wedge^2 \mathbb{R}^5$ . This is achieved in terms of the operator

$$Y_{ijkl} = 4\Upsilon_{ijm} \Upsilon_{klp} g^{mp}.$$

This, viewed as an endomorphism of  $\otimes^2 \mathbb{R}^5$  given by

$$Y(w)_{ik} = g^{mj} g^{pl} Y_{ijkl} w_{mp},$$

has the following eigenspaces:

$$\begin{aligned} \odot_1 &= \{ S \in \otimes^2 \mathbb{R}^5 \mid Y(S) = 14 \cdot S \} = \{ S = \lambda \cdot g, \lambda \in \mathbb{R} \}, \\ \wedge_3 &= \{ F \in \otimes^2 \mathbb{R}^5 \mid Y(F) = 7 \cdot F \} = \mathfrak{sl}(2, \mathbb{R}), \\ \odot_5 &= \{ S \in \otimes^2 \mathbb{R}^5 \mid Y(S) = -3 \cdot S \}, \\ \wedge_7 &= \{ F \in \otimes^2 \mathbb{R}^5 \mid Y(F) = -8 \cdot F \}, \\ \odot_9 &= \{ S \in \otimes^2 \mathbb{R}^5 \mid Y(S) = 4 \cdot S \}. \end{aligned}$$

Here the index  $k$  in  $\odot_k$  or  $\wedge_k$  denotes the dimension of the eigenspace.

The decomposition

$$(3.16) \quad \otimes^2 \mathbb{R}^5 = \odot_1 \oplus \odot_5 \oplus \odot_9 \oplus \wedge_3 \oplus \wedge_7$$

is  $\mathbf{GL}(2, \mathbb{R})$  invariant. All the components in this decomposition are  $\mathbf{GL}(2, \mathbb{R})$ -irreducible. We have the following

**Proposition 3.8.** *Under the action of  $\mathbf{GL}(2, \mathbb{R})$  the irreducible components of  $\wedge^3 \mathbb{R}^5 = * \wedge^2 \mathbb{R}^5$  are*

$$\wedge^3 \mathbb{R}^5 = \wedge_3 \oplus \wedge_7.$$

At this stage an interesting question arises: Can we give examples of nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  structures whose characteristic connection has torsion of a ‘pure’ type  $T_{ijk} \in \Lambda_3$ ?

In section 5 we give an affirmative answer to this question. Here we only state a useful

**Lemma 3.9.** *The 3-dimensional vector space  $\Lambda_3$ , when expressed in terms of an adapted coframe  $\theta^i$  of Definition 3.2 is*

$$\Lambda_3 = \text{Span}_{\mathbb{R}} \left\{ \theta^0 \wedge \theta^3 - 3\theta^1 \wedge \theta^2, \theta^0 \wedge \theta^4 - 2\theta^1 \wedge \theta^3, \theta^1 \wedge \theta^4 - 3\theta^2 \wedge \theta^3 \right\}.$$

Similarly, in an adapted coframe  $\theta^i$ , the Hodge dual  $*\Lambda_3$  of  $\Lambda_3$  is

$$\begin{aligned} *\Lambda_3 = \text{Span}_{\mathbb{R}} \left\{ -\theta^0 \wedge \theta^1 \wedge \theta^4 + 2\theta^0 \wedge \theta^2 \wedge \theta^3, -\theta^0 \wedge \theta^2 \wedge \theta^4 + 8\theta^1 \wedge \theta^2 \wedge \theta^3, \right. \\ \left. -\theta^0 \wedge \theta^3 \wedge \theta^4 + 2\theta^1 \wedge \theta^2 \wedge \theta^4 \right\}. \end{aligned}$$

In particular, torsion  $T_{jk}^i$  of the characteristic connection  $\Gamma$  in system (3.15) is of pure type in  $\Lambda_3$  if and only if, in an adapted coframe  $\theta^i$ , we have  $g_{il}T_{jk}^l = T_{[ijk]}$ , and its corresponding 3-form  $T = \frac{1}{6}g_{il}T_{jk}^l\theta^i \wedge \theta^j \wedge \theta^k \in *\Lambda_3$ .

Now we pass to the analysis of the curvature. The curvature tensor  $R_{jkl}^i$  of a  $\mathfrak{gl}(2, \mathbb{R})$  connection defines the following objects:

$$\begin{aligned} R_{ij} &= R_{ikj}^k && \text{the Ricci tensor,} \\ R &= R_{ij}g^{ij} && \text{the Ricci scalar,} \\ R_{\mathbf{v}}^i &= \Upsilon^{ijk}R_{jk} && \text{the Ricci vector,} \\ (dA)_{ij} &= \frac{2}{5}R_{kij}^k && \text{the Maxwell 2-form.} \end{aligned}$$

The Ricci tensor belongs to the space  $\otimes^2 \mathbb{R}^5$  and decomposes according to (3.16). The Ricci symmetric tensor reads

$$(3.17) \quad R_{(ij)} = \frac{1}{5}Rg_{ij} + \frac{2}{7}R_{\mathbf{v}}^k\Upsilon_{ijk} + R_{ij}^{(9)},$$

where  $\frac{1}{5}Rg_{ij}$  is its  $\odot_1$  part,  $\frac{2}{7}R_{\mathbf{v}}^k\Upsilon_{ijk}$  is its  $\odot_5$  part and  $R_{ij}^{(9)}$  is its  $\odot_9$  part defined by (3.17). The antisymmetric Ricci tensor decomposes into

$$R_{[ij]} = R_{ij}^{(3)} + R_{ij}^{(7)}$$

with the respective  $\Lambda_3$  and  $\Lambda_7$  components given by

$$\begin{aligned} R_{ij}^{(3)} &= \frac{8}{15}R_{[ij]} + \frac{1}{15}Y(R_{[ ]})_{ij}, \\ R_{ij}^{(7)} &= \frac{7}{15}R_{[ij]} - \frac{1}{15}Y(R_{[ ]})_{ij}. \end{aligned}$$

Here  $Y(R_{[ ]})$  denotes the value of the operator  $Y$  on  $R_{[ij]}$ . Likewise, for the Maxwell form we have

$$(dA)_{ij} = dA_{ij}^{(3)} + dA_{ij}^{(7)}$$

and

$$\begin{aligned} dA_{ij}^{(3)} &= \frac{8}{15}(dA)_{ij} + \frac{1}{15}Y(dA)_{ij}, \\ dA_{ij}^{(7)} &= \frac{7}{15}(dA)_{ij} - \frac{1}{15}Y(dA)_{ij}. \end{aligned}$$

The Ricci tensor and the Maxwell 2-form have  $25 + 10 = 35$  coefficients out of total number of 40 coefficients of the curvature. Since, c.f. [14],

$$\mathfrak{gl}(2, \mathbb{R}) \otimes \bigwedge^2 \mathbb{R}^5 = \odot_1 \oplus 2\bigwedge_3 \oplus 2\odot_5 \oplus 2\bigwedge_7 \oplus \odot_9,$$

the remaining 5 parameters are related to the coefficients of a vector field  $K^m$ , which is independent of the Ricci tensor. It is defined in terms of the totally skew symmetric part of the curvature. Using the volume form  $\eta^{ijklm}$ , we have

$$K^m = R_{ijkl} \eta^{ijklm},$$

and the so defined  $K^m$  yields the missing five components of the curvature. Thus we have the following

**Proposition 3.10.** *The irreducible components of the curvature  $R_{ijkl}$  of a characteristic connection are given by*

$$R, \quad R_{\mathbf{v}}^i, \quad R_{ij}^{(9)}, \quad R_{ij}^{(3)}, \quad R_{ij}^{(7)}, \quad dA_{ij}^{(3)}, \quad dA_{ij}^{(7)}, \quad K^i.$$

It is interesting to ask what is the decomposition of the curvature if the characteristic connection has torsion in three-dimensional representation  $\bigwedge_3$ . It appears tha it has a very special algebraic form. Writing the structural equations for a characteristic connection with the torsion in  $\bigwedge_3$

$$\begin{aligned} T = & \frac{1}{12} t_1 (-\theta^0 \wedge \theta^1 \wedge \theta^4 + 2\theta^0 \wedge \theta^2 \wedge \theta^3) + \\ & + \frac{1}{12} t_2 (-\theta^0 \wedge \theta^2 \wedge \theta^4 + 8\theta^1 \wedge \theta^2 \wedge \theta^3) + \\ & + \frac{1}{4} t_3 (-\theta^0 \wedge \theta^3 \wedge \theta^4 + 2\theta^1 \wedge \theta^2 \wedge \theta^4) \end{aligned}$$

and utilising Bianchi identites, we get the following

**Proposition 3.11.** *Let  $\Gamma$  be a characteristic connection of a  $\mathbf{GL}(2, \mathbb{R})$  structure with torsion in  $\bigwedge_3$  given above. Then*

- The Ricci tensor component  $R_{ij}^{(9)} = 0$ , which means that

$$R_{(ij)} = \frac{1}{5} R g_{ij} + \frac{2}{7} R_{\mathbf{v}}^k \Upsilon_{ijk}.$$

- The skew symmetric Ricci tensor and the Maxwell 2-form are related by

$$dA^{(3)} = 4R^{(3)}, \quad dA^{(7)} = \frac{2}{3} R^{(7)}.$$

- The Ricci vector  $R_{\mathbf{v}}$  is fully determined by  $T$ :

$$\begin{aligned} R_{\mathbf{v}}^i = & (40)^2 (*T)_{jk} (*T)_{lm} g^{kl} \Upsilon^{jmi} \\ = & \frac{7}{6} \sqrt{3} \left( t_3^2, -\frac{1}{3} t_2 t_3, \frac{1}{9} t_1 t_3 + \frac{2}{27} t_2^2, -\frac{1}{9} t_1 t_2, \frac{1}{9} t_1^2 \right). \end{aligned}$$

Thus, the curvature is fully described by  $T, R, dA^{(3)}, dA_{ij}^{(7)}$  and  $K^i$ .

**3.2. Arbitrary  $\mathbf{GL}(2, \mathbb{R})$  structures.** So far we have been able to introduce a unique  $\mathbf{GL}(2, \mathbb{R})$ -valued connection for a nearly integrable  $(M^5, [g, \Upsilon, A])$  only. Nevertheless such a connection can be always introduced. To see this consider a  $\mathbf{GL}(2, \mathbb{R})$ -invariant conformal pairing in  $\mathfrak{co}(3, 2) \otimes \mathbb{R}^5$  given by

$$(\overset{w}{\Gamma}, \overset{w}{\Gamma}') = g^{il} g^{jm} g^{kp} \overset{w}{\Gamma}_{ijk} \overset{w}{\Gamma}'_{lmp},$$

where  $\tilde{\Gamma}, \tilde{\Gamma}' \in \mathfrak{co}(3, 2) \otimes \mathbb{R}^5$ . We use the orthogonal complement of  $\ker \tilde{\Upsilon} \subset \mathfrak{co}(3, 2) \otimes \mathbb{R}^5$  with respect to this pairing:

$$\ker \tilde{\Upsilon}^\perp = \{\tilde{\Gamma} \in \mathfrak{co}(3, 2) \otimes \mathbb{R}^5 \text{ s.t. } (\ker \tilde{\Upsilon}, \tilde{\Gamma}) = 0\}.$$

This vector space is 30-dimensional. It contains a 5-dimensional subspace spanned by  $g_{ij}A_m$ , which is related to the  $\mathbb{R}$  factor in the split  $\mathfrak{gl}(2, \mathbb{R}) = \mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{co}(3, 2) = \mathbb{R} \oplus \mathfrak{so}(3, 2)$ . Thus it is reasonable to consider the intersection, say  $V_{25}$ , of this 30-dimensional space with  $\mathfrak{so}(3, 2) \otimes \mathbb{R}^5$ . This 25-dimensional space

$$V_{25} = \ker \tilde{\Upsilon}^\perp \cap (\mathfrak{so}(3, 2) \otimes \mathbb{R}^5)$$

has, in turn, zero intersection with  $(\mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5) \oplus \bigwedge^3 \mathbb{R}^5$  and provides the  $\mathbf{GL}(2, \mathbb{R})$  invariant decomposition of  $\mathfrak{co}(3, 2) \otimes \mathbb{R}^5$ :

$$\mathfrak{co}(3, 2) \otimes \mathbb{R}^5 = (\mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5) \oplus \bigwedge^3 \mathbb{R}^5 \oplus V_{25}.$$

Therefore, if we choose a coframe adapted to a representative  $(g, \Upsilon, A)$  we can uniquely decompose the Weyl connection coefficients  $\tilde{\Gamma}_{ijk}^w \in \mathfrak{co}(3, 2) \otimes \mathbb{R}^5$  of our *arbitrary*  $\mathbf{GL}(2, \mathbb{R})$  structure according to

$$\tilde{\Gamma}_{ijk}^w = \Gamma_{ijk} + \frac{1}{2}B_{ijk}.$$

Now  $\Gamma_{ijk} \in \mathfrak{gl}(2, \mathbb{R}) \otimes \mathbb{R}^5$ , and they are interpreted as new connection coefficients; the tensor  $B_{ijk}$  belongs to  $\bigwedge^3 \mathbb{R}^5 \oplus V_{25}$  and its antisymmetrization  $T_{ijk} = B_{i[jk]}$  is now interpreted as the torsion of  $\Gamma$ . Thus, *every*  $\mathbf{GL}(2, \mathbb{R})$  structure  $(M^5, [g, \Upsilon, A])$  uniquely defines a  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection with torsion in  $\bigwedge^3 \mathbb{R}^5 \oplus V_{25}$ . The torsion is not totally skew anymore. Space  $V_{25}$  further decomposes onto the  $\mathbf{GL}(2, \mathbb{R})$ -irreducible components according to  $V_{25} = \odot_5 \oplus \odot_9 \oplus \odot_{11}$ . The  $\mathbf{GL}(2, \mathbb{R})$  structures equipped with the unique  $\mathfrak{gl}(2, \mathbb{R})$  connection which has torsion in  $V_{25}$  find application in the theory of integrable equations of hydrodynamic type [11].

#### 4. $\mathbf{GL}(2, \mathbb{R})$ BUNDLE

*First*, we describe an irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure  $[g, \Upsilon, A]$  on  $M^5$  in the language of principal bundles.

Every irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure  $[g, \Upsilon, A]$  on a 5-manifold  $M^5$  defines the 9-dimensional bundle  $\mathbf{GL}(2, \mathbb{R}) \rightarrow P \rightarrow M^5$ , the  $\mathbf{GL}(2, \mathbb{R})$  reduction of the bundle of linear frames  $\mathbf{GL}(5, \mathbb{R}) \rightarrow F(M^5) \rightarrow M^5$ . If  $[g, \Upsilon, A]$  is equipped with a  $\mathfrak{gl}(2, \mathbb{R})$  connection  $\Gamma$  then the structural equations on  $M^5$  read

$$\begin{aligned} d\omega^i + \Gamma^i_j \wedge \omega^j &= \frac{1}{2}T^i_{jk}\omega^j \wedge \omega^k, \\ d\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j &= \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l. \end{aligned}$$

Here  $(\omega^i)$  is an adapted coframe and  $\Gamma = (\Gamma^i_j)$  is written in the representation (2.5). We lift these structural equations to  $P$  obtaining:

$$\begin{aligned} d\theta^0 &= 4(\Gamma_1 + \Gamma_0) \wedge \theta^0 - 4\Gamma_+ \wedge \theta^1 + \frac{1}{2}T^0_{ij}\theta^i \wedge \theta^j, \\ d\theta^1 &= -\Gamma_- \wedge \theta^0 + (4\Gamma_1 + 2\Gamma_0) \wedge \theta^1 - 3\Gamma_+ \wedge \theta^2 + \frac{1}{2}T^1_{ij}\theta^i \wedge \theta^j, \\ d\theta^2 &= -2\Gamma_- \wedge \theta^1 + 4\Gamma_1 \wedge \theta^2 - 2\Gamma_+ \wedge \theta^3 + \frac{1}{2}T^2_{ij}\theta^i \wedge \theta^j, \\ d\theta^3 &= -3\Gamma_- \wedge \theta^2 + (4\Gamma_1 - 2\Gamma_0) \wedge \theta^3 - \Gamma_+ \wedge \theta^4 + \frac{1}{2}T^3_{ij}\theta^i \wedge \theta^j, \\ (4.1) \quad d\theta^4 &= -4\Gamma_- \wedge \theta^3 + 4(\Gamma_1 - \Gamma_0) \wedge \theta^4 + \frac{1}{2}T^4_{ij}\theta^i \wedge \theta^j, \end{aligned}$$

$$\begin{aligned}
d\Gamma_+ &= 2\Gamma_0 \wedge \Gamma_+ + \frac{1}{2}R_{+ij}\theta^i \wedge \theta^j, \\
d\Gamma_- &= -2\Gamma_0 \wedge \Gamma_- + \frac{1}{2}R_{-ij}\theta^i \wedge \theta^j, \\
d\Gamma_0 &= \Gamma_+ \wedge \Gamma_- + \frac{1}{2}R_{0ij}\theta^i \wedge \theta^j, \\
d\Gamma_1 &= \frac{1}{2}R_{1ij}\theta^i \wedge \theta^j,
\end{aligned}$$

with the forms  $\theta^i$  being the components of the canonical  $\mathbb{R}^5$ -valued form  $\theta$  on  $P$ , c.f. [13]. In a coordinate system  $(x, a)$  on  $P$ ,  $x \in M^5$ ,  $a \in \mathbf{GL}(2, \mathbb{R})$ , which is compatible with the local trivialisation  $P \cong M^5 \times \mathbf{GL}(2, \mathbb{R})$  they are given by

$$\theta^i(x, a) = (a^{-1})^i_j \omega^j(x).$$

The connection forms  $(\Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$  are defined in terms of (2.6) via

$$\Gamma_-(E_-)^i_j + \Gamma_+(E_+)^i_j + \Gamma_0(E_0)^i_j + \Gamma_1(E_1)^i_j = (a^{-1})^i_k \Gamma_l^k(x) a^l_j + (a^{-1})^i_k da^k_j.$$

Note that  $(\theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$  is a coframe on  $P$  and the class of 1-forms  $[A]$  lifts to a 1-form  $\tilde{A} = -8\Gamma_1$ .

*Second*, we change the point of view. Suppose that we *are given* a nine dimensional manifold  $P$  equipped with a coframe of nine 1-forms  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$  on it. Suppose that these linearly independent forms, together with some functions  $T_{jk}^i, R_{ijk}^l$ , satisfy the system (4.1) on  $P$ . What we can say about such a 9-dimensional manifold  $P$ ?

To answer this question consider a distribution  $\mathfrak{h}$  on  $P$  which annihilates the forms  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$ :

$$\mathfrak{h} = \{X \in TP \text{ s.t. } X \lrcorner \theta^i = 0, \ i = 0, 1, 2, 3, 4\}.$$

Then the first five equations of the system (4.1) guarantee that the forms  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$  satisfy the Fröbenius condition,

$$d\theta^i \wedge \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0, \quad \forall i = 0, 1, 2, 3, 4$$

and that, in turn, the distribution  $\mathfrak{h}$  is integrable. Thus manifold  $P$  is foliated by 4-dimensional leaves tangent to the distribution  $\mathfrak{h}$ .

Now on  $P$  we consider two multilinear symmetric forms. The bilinear one, defined by

$$(4.2) \quad \tilde{g} = \theta^0\theta^4 - 4\theta^1\theta^3 + 3(\theta^2)^2,$$

and the three-linear one given by

$$(4.3) \quad \tilde{\Upsilon} = 3\sqrt{3}(\theta^0\theta^2\theta^4 + 2\theta^1\theta^2\theta^3 - (\theta^2)^3 - \theta^0(\theta^3)^2 - \theta^4(\theta^1)^2).$$

Of course, since the 1-forms  $(\Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$  are not present in the definitions (4.2), (4.3), then  $\tilde{g}$  and  $\tilde{\Upsilon}$  are *degenerate*. For example, the signature of the bilinear form  $\tilde{g}$  is  $(+, +, +, -, -, 0, 0, 0, 0)$ . The degenerate directions for these two forms are just the directions tangent to the leaves of the foliation generated by  $\mathfrak{h}$ . Let us denote by  $(X_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$  the frame of vector fields on  $P$  dual to the 1-forms  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$ . In particular  $(X_5, X_6, X_7, X_8)$  constitutes a basis for  $\mathfrak{h}$ , and we have  $X_\mu \lrcorner \theta^i = 0$  for each  $\mu = 5, 6, 7, 8$  and  $i = 0, 1, 2, 3, 4$ . Using this, and the exterior derivatives of  $\theta^i$  given in the first five equations (4.1), we easily find the Lie derivatives of  $\tilde{g}$  and  $\tilde{\Upsilon}$  along the directions tangent to the leaves of  $\mathfrak{h}$ . These are:

$$\mathcal{L}_{X_\mu} \tilde{g} = 8(X_\mu \lrcorner \Gamma_1) \tilde{g}, \quad \mathcal{L}_{X_\mu} \tilde{\Upsilon} = 12(X_\mu \lrcorner \Gamma_1) \tilde{\Upsilon}, \quad \forall \mu = 5, 6, 7, 8.$$

Moreover, if we denote

$$(4.4) \quad \tilde{A} = -8\Gamma_1,$$

and we use the last of equations (4.1), we also find that

$$\mathcal{L}_{X_\mu} \tilde{A} = -8d(X_\mu \lrcorner \Gamma_1), \quad \forall \mu = 5, 6, 7, 8.$$

This is enough to deduce that the objects  $(\tilde{g}, \tilde{\Upsilon}, \tilde{A})$  descend to the 5-dimensional leaf space  $M^5 = P/\mathfrak{h}$ . There they define a conformal class of triples  $(g, \Upsilon, A)$  with the transformation rules  $g \rightarrow e^{2\phi}g$ ,  $\Upsilon \rightarrow e^{3\phi}\Upsilon$ ,  $A \rightarrow A - 2d\phi$ . Due to the fact that, when passing to the quotient  $M^5 = P/\mathfrak{h}$ , we reduced the degenerate directions of  $\tilde{g}$  and  $\tilde{\Upsilon}$  to points of  $M^5$ , the resulting descended triples  $(g, \Upsilon, A)$  have non-degenerate  $g$  of signature  $(3, 2)$  and non-degenerate  $\Upsilon$ . It is clear that together with  $A$  they define an irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure on  $M^5$ : a section  $s: M^5 \rightarrow P$  is an adapted coframe on  $M^5$ , the triple  $(s^*\tilde{g}, s^*\tilde{\Upsilon}, s^*\tilde{A})$  is a representative of the structure, the forms  $s^*\Gamma_-, s^*\Gamma_+, s^*\Gamma_0, s^*\Gamma_1$  are  $\mathfrak{gl}(2, \mathbb{R})$  connection 1-forms on  $M^5$  and  $s^*T, s^*R$  are torsion and curvature of this connection, respectively. We have the following

**Proposition 4.1.** *Every 9-dimensional manifold  $P$  equipped with nine 1-forms  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$  which*

- *are linearly independent at every point of  $P$ ,*
- *satisfy system (4.1) with some functions  $T^i_{jk}, R^i_{jkl}$  on  $P$ ,*

*is foliated by 4-dimensional leaves over a 5-dimensional space  $M^5$ , which is the base for the fibration  $P \rightarrow M^5$ . The manifold  $M^5$  is equipped with a natural irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure  $[g, \Upsilon, A]$  and a  $\mathfrak{gl}(2, \mathbb{R})$  connection compatible with it. The torsion and the curvature of this connection is given by  $T^i_{jk}$  and  $R^i_{jkl}$ .*

## 5. 5TH ORDER ODE AS NEARLY INTEGRABLE $\mathbf{GL}(2, \mathbb{R})$ GEOMETRY WITH ‘SMALL’ TORSION. MAIN THEOREM

A large number of examples of nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  structures in dimension five is related to 5th order ODEs. This is mainly due to the following, well known,

**Proposition 5.1.** *An ordinary differential equation  $y^{(5)} = 0$  has  $\mathbf{GL}(2, \mathbb{R}) \times_{\rho_5} \mathbb{R}^5$  as its group of contact symmetries. Here  $\rho_5: \mathbf{GL}(2, \mathbb{R}) \rightarrow \mathbf{GL}(5, \mathbb{R})$  is the 5-dimensional irreducible representation of  $\mathbf{GL}(2, \mathbb{R})$ .*

To explain the above statement we consider a general 5th order ODE

$$(5.1) \quad y^{(5)} = F(x, y, y', y'', y^{(3)}, y^{(4)})$$

for a real function  $\mathbb{R} \ni x \mapsto y(x) \in \mathbb{R}$ . Let us introduce the notation  $y_1 = y'$ ,  $y_2 = y''$ ,  $y_3 = y^{(3)}$ ,  $y_4 = y^{(4)}$  and  $F_i = \frac{\partial F}{\partial y_i}$ ,  $i = 1, 2, 3, 4$ ,  $F_y = \frac{\partial F}{\partial y}$ . The functions  $(x, y, y_1, y_2, y_3, y_4)$  form a local coordinate system in the 4-order jet space  $J$  of curves in  $\mathbb{R}^2$ . Define a total derivative, which is a vector field in  $J$

$$(5.2) \quad \mathcal{D} = \partial_x + y_1 \partial_y + y_2 \partial_{y_1} + y_3 \partial_{y_2} + y_4 \partial_{y_3} + F \partial_{y_4}.$$

With the help of  $\mathcal{D}$  the derivatives are given by formulae  $y_1 = \mathcal{D}y/\mathcal{D}x$ ,  $y_2 = \mathcal{D}y_1/\mathcal{D}x$  and so on, up to  $y_5 = \mathcal{D}y_4/\mathcal{D}x$ .

A contact transformation of variables in a 5-order ODE is a transformation that mixes the independent variable  $x$ , the dependent variable  $y$  and the first derivative  $y_1$  in such a way that the meaning of the first derivative is retained:

**Definition 5.2.** A contact transformation of variables is an invertible, sufficiently smooth transformation of the form

$$(5.3) \quad \begin{pmatrix} x \\ y \\ y_1 \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{y}_1 \end{pmatrix} = \begin{pmatrix} \bar{x}(x, y, y_1) \\ \bar{y}(x, y, y_1) \\ \bar{y}_1(x, y, y_1) \end{pmatrix}$$

satisfying the condition

$$\bar{y}_1 = \frac{\mathcal{D}\bar{y}}{\mathcal{D}\bar{x}}. \quad (\text{preservation of first derivative})$$

The higher order derivatives are given by the iterative formula

$$y_{n+1} \mapsto \bar{y}_{n+1} = \frac{\mathcal{D}\bar{y}_n}{\mathcal{D}\bar{x}}, \quad i = 1, 2, 3, 4.$$

Let us now consider the equation  $y^{(5)} = 0$ . We show how the flat torsionfree 5-dimensional irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure is naturally generated on its space of solutions by means of the symmetry group. A solution to  $y^{(5)} = 0$  is of the form

$$(5.4) \quad y(x) = c_4 x^4 + 4c_3 x^3 + 6c_2 x^2 + 4c_1 x + c_0$$

with five integration constants  $c_0, c_1, c_2, c_3, c_4$ . Then a solution of  $y^{(5)} = 0$  may be identified with a point  $c = (c_0, c_1, c_2, c_3, c_4)^T$  in  $\mathbb{R}^5$ . A contact symmetry of  $y^{(5)} = 0$  is a contact transformation of variables that transforms its solutions into solutions. Group of contact symmetries of  $y^{(5)} = 0$  is generated by the following one-parameter groups of transformations on the  $xy$ -plane:

$$\begin{aligned} \varphi_t^0(x, y) &= (x, y + t), & \varphi_t^1(x, y) &= (x, y + 4xt), \\ \varphi_t^2(x, y) &= (x, y + 6x^2t), & \varphi_t^3(x, y) &= (x, y + 4x^3t), \\ \varphi_t^4(x, y) &= (x, y + x^4t), & \varphi_t^5(x, y) &= (xe^{2t}, ye^{4t}), \\ \varphi_t^6(x, y) &= (x, ye^{4t}), & \varphi_t^7(x, y) &= (x + t, y), \\ \varphi_t^8(x, y) &= \left( \frac{x}{1 + xt}, \frac{y}{(1 + xt)^4} \right) \end{aligned}$$

and the transformation rules for  $y_1$  are given by  $\varphi^A(y_1) = \mathcal{D}(\varphi^A(y))/\mathcal{D}(\varphi^A(x))$ ,  $A = 0, \dots, 8$ .

Transforming (5.4) according to the above formulae we find that  $\varphi_t^0, \dots, \varphi_t^4$  are translations in the space of solutions:

$$\varphi_t^0(c) = (c_0 - t, c_1, c_2, c_3, c_4)^T, \dots, \quad \varphi_t^4(c) = (c_0, c_1, c_2, c_3, c_4 - t)^T,$$

while transformations  $\varphi_t^5, \dots, \varphi_t^8$  generate  $\mathbf{GL}(2, \mathbb{R})$  and act through the 5-dimensional irreducible representation (2.6):

$$\begin{aligned} \varphi_t^5(c) &= \exp(tE_0)c, & \varphi_t^6(c) &= \exp(tE_1)c, \\ \varphi_t^7(c) &= \exp(tE_+)c, & \varphi_t^8(c) &= \exp(tE_-)c. \end{aligned}$$

Of course,  $\mathbf{GL}(2, \mathbb{R})$  stabilizes the origin  $(0, 0, 0, 0, 0)$  in  $\mathbb{R}^5$ , thus the space of solutions is the homogeneous space  $\mathbf{GL}(2, \mathbb{R}) \rightarrow \mathbf{GL}(2, \mathbb{R}) \times_{\rho_5} \mathbb{R}^5 \rightarrow \mathbb{R}^5$ . The total space of this bundle is equipped with the Maurer – Cartan form  $\omega_{MC}$  of  $\mathbf{GL}(2, \mathbb{R}) \times_{\rho_5} \mathbb{R}^5$ .

Choosing an appropriate basis in  $\mathfrak{gl}(2, \mathbb{R})$  and writing explicitly the structural equations  $d\omega_{MC} + \omega_{MC} \wedge \omega_{MC} = 0$  we get

$$\begin{aligned} d\theta^0 &= 4(\Gamma_1 + \Gamma_0) \wedge \theta^0 - 4\Gamma_+ \wedge \theta^1, \\ d\theta^1 &= -\Gamma_- \wedge \theta^0 + (4\Gamma_1 + 2\Gamma_0) \wedge \theta^1 - 3\Gamma_+ \wedge \theta^2, \\ d\theta^2 &= -2\Gamma_- \wedge \theta^1 + 4\Gamma_1 \wedge \theta^2 - 2\Gamma_+ \wedge \theta^3, \\ d\theta^3 &= -3\Gamma_- \wedge \theta^2 + (4\Gamma_1 - 2\Gamma_0) \wedge \theta^3 - \Gamma_+ \wedge \theta^4, \\ d\theta^4 &= -4\Gamma_- \wedge \theta^3 + 4(\Gamma_1 - \Gamma_0) \wedge \theta^4, \\ d\Gamma_+ &= 2\Gamma_0 \wedge \Gamma_+, \\ d\Gamma_- &= -2\Gamma_0 \wedge \Gamma_-, \\ d\Gamma_0 &= \Gamma_+ \wedge \Gamma_-, \\ d\Gamma_1 &= 0, \end{aligned}$$

which is the system (4.1) with all the torsion and curvature coefficients equal to zero. According to proposition 4.1 it yields a flat and torsionfree irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure on the space of solutions of  $y^{(5)} = 0$ . Again, as in the case of the algebraic geometric realization of section 2, we learned about that from E. V. Ferapontow [10].

We now pass to a more general situation, namely to the equation (5.1) with a general  $F$ . The following questions are in order:

What shall one assume about  $F$  to be able to construct an irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure on the solution space of the corresponding ODE? Is the case  $F = 0$  very special, or there are other ODEs, contact nonequivalent to the  $F = 0$  case, which define a  $\mathbf{GL}(2, \mathbb{R})$  geometry on the solution space? If the answer is affirmative, how do we find such  $F$ s and what can we say about the corresponding  $\mathbf{GL}(2, \mathbb{R})$  structures?

Answer to these questions is given by the following

**Theorem 5.3** (Main theorem). *Every contact equivalence class of 5th order ODEs satisfying the Wünschmann conditions*

$$\begin{aligned} 50D^2F_4 - 75DF_3 + 50F_2 - 60F_4DF_4 + 30F_3F_4 + 8F_4^3 &= 0, \\ 375D^2F_3 - 1000DF_2 + 350DF_4^2 + 1250F_1 - 650F_3DF_4 + 200F_3^2 - \\ 150F_4DF_3 + 200F_2F_4 - 140F_4^2DF_4 + 130F_3F_4^2 + 14F_4^4 &= 0, \\ (5.5) \quad 1250D^2F_2 - 6250DF_1 + 1750DF_3DF_4 - 2750F_2DF_4 - \\ 875F_3DF_3 + 1250F_2F_3 - 500F_4DF_2 + 700(DF_4)^2F_4 + \\ 1250F_1F_4 - 1050F_3F_4DF_4 + 350F_3^2F_4 - 350F_4^2DF_3 + \\ 550F_2F_4^2 - 280F_4^3DF_4 + 210F_3F_4^3 + 28F_4^5 + 18750F_y &= 0 \end{aligned}$$

defines a nearly integrable irreducible  $\mathbf{GL}(2, \mathbb{R})$  geometry  $(M^5, [g, \Upsilon, A])$  on the space  $M^5$  of its solutions. This geometry has the characteristic connection with torsion  $T$  of the ‘pure’ type in the 3-dimensional irreducible representation  $\wedge_3$ . The first structural equation for this connection are the following:

$$d\theta^0 = 4(\Gamma_1 + \Gamma_0) \wedge \theta^0 - 4\Gamma_+ \wedge \theta^1 +$$



$$\begin{aligned}
& -\frac{1}{3}t_1\theta^0 \wedge \theta^1 - \frac{1}{3}t_2\theta^0 \wedge \theta^2 - t_3\theta^0 \wedge \theta^3 + 2t_3\theta^1 \wedge \theta^2, \\
d\theta^1 &= -\Gamma_- \wedge \theta^0 + (4\Gamma_1 + 2\Gamma_0) \wedge \theta^1 - 3\Gamma_+ \wedge \theta^2 + \\
& -\frac{1}{6}t_1\theta^0 \wedge \theta^2 - \frac{1}{4}t_3\theta^0 \wedge \theta^4 - \frac{2}{3}t_2\theta^1 \wedge \theta^2, \\
d\theta^2 &= -2\Gamma_- \wedge \theta^1 + 4\Gamma_1 \wedge \theta^2 - 2\Gamma_+ \wedge \theta^3 + \\
& -\frac{1}{9}t_1\theta^0 \wedge \theta^3 + \frac{1}{18}t_2\theta^0 \wedge \theta^4 - \frac{4}{9}t_2\theta^1 \wedge \theta^3 - \frac{1}{3}t_3\theta^1 \wedge \theta^4, \\
d\theta^3 &= -3\Gamma_- \wedge \theta^2 + (4\Gamma_1 - 2\Gamma_0) \wedge \theta^3 - \Gamma_+ \wedge \theta^4 + \\
& + \frac{1}{12}t_1\theta^0 \wedge \theta^4 - \frac{2}{3}t_2\theta^2 \wedge \theta^3 - \frac{1}{2}t_3\theta^2 \wedge \theta^4, \\
d\theta^4 &= -4\Gamma_- \wedge \theta^3 + 4(\Gamma_1 - \Gamma_0) \wedge \theta^4 + \\
& -\frac{1}{3}t_1\theta^1 \wedge \theta^4 + \frac{2}{3}t_1\theta^2 \wedge \theta^3 - \frac{1}{3}t_2\theta^2 \wedge \theta^4 - t_3\theta^3 \wedge \theta^4
\end{aligned}$$

with the torsion coefficients

$$\begin{aligned}
t_3 &= \frac{6(\alpha_5^5)^2}{5\alpha_1^1} F_{44}, \\
t_2 &= \frac{9\alpha_5^5}{50(\alpha_1^1)^2} [\alpha_1^1(10\mathcal{D}F_{44} + 3F_4F_{44}) + 5\alpha_0^1F_{44}], \\
t_1 &= [1000(\alpha_1^1)^3]^{-1} \times \\
& \left( 225(\alpha_0^1)^2F_{44} + 90\alpha_0^1\alpha_1^1(10\mathcal{D}F_{44} + 3F_4F_{44}) + \right. \\
& -9(\alpha_1^1)^2[20(5\mathcal{D}F_{34} + 20F_{24} - 15F_{33} + 3F_4\mathcal{D}F_{44} - 11F_4F_{34}) + \\
& \left. + F_{44}(-120\mathcal{D}F_4 + 340F_3 + 51F_4^2)] \right),
\end{aligned}$$

where  $(y, y_1, y_2, y_3, y_4, x, \alpha_1^1, \alpha_0^1, \alpha_5^5)$  is a local coordinate system on  $\mathbf{GL}(2, \mathbb{R}) \rightarrow P \rightarrow M^5$ . The second structural equations are the following:

$$\begin{aligned}
d\Gamma_+ &= 2\Gamma_0 \wedge \Gamma_+ + \left(\frac{1}{6}b_2 - \frac{1}{81}t_1^2 + \frac{5}{3}c_5\right)\theta_0 \wedge \theta_1 + \left(-\frac{2}{81}t_1t_2 - \frac{10}{3}c_4 + \frac{5}{12}b_3\right)\theta_0 \wedge \theta_2 + \\
& + \left(-\frac{1}{243}t_2^2 - \frac{1}{162}t_1t_3 + \frac{10}{3}c_3 - \frac{1}{30}R + b_4 - \frac{1}{4}a_2\right)\theta_0 \wedge \theta_3 + \\
& + \left(\frac{1}{54}t_2t_3 - \frac{1}{8}a_3 - \frac{5}{3}c_2 + \frac{1}{12}b_5\right)\theta_0 \wedge \theta_4 + \\
& + \left(-\frac{1}{27}t_2^2 - \frac{1}{18}t_1t_3 + \frac{1}{10}R + 2b_4 + \frac{3}{4}a_2\right)\theta_1 \wedge \theta_2 + \\
& + \left(-\frac{1}{9}t_2t_3 + \frac{1}{4}a_3 + \frac{2}{3}b_5\right)\theta_1 \wedge \theta_3 + \left(\frac{1}{18}t_3^2 + \frac{5}{3}c_1 + \frac{1}{6}b_6\right)\theta_1 \wedge \theta_4 + \\
& + \left(-\frac{5}{18}t_3^2 - \frac{10}{3}c_1 + \frac{1}{3}b_6\right)\theta_2 \wedge \theta_3 + \frac{1}{4}b_7\theta_2 \wedge \theta_4, \\
d\Gamma_- &= -2\Gamma_0 \wedge \Gamma_- + \frac{1}{4}b_1\theta_0 \wedge \theta_2 + \left(\frac{1}{6}b_2 - \frac{1}{162}t_1^2 - \frac{5}{3}c_5\right)\theta_0 \wedge \theta_3 + \\
& + \left(-\frac{1}{162}t_1t_2 + \frac{5}{3}c_4 + \frac{1}{12}b_3 - \frac{1}{8}a_1\right)\theta_0 \wedge \theta_4 + \\
& + \left(\frac{5}{162}t_1^2 + \frac{1}{3}b_2 + \frac{10}{3}c_5\right)\theta_1 \wedge \theta_2 + \left(\frac{1}{27}t_1t_2 + \frac{2}{3}b_3 + \frac{1}{4}a_1\right)\theta_1 \wedge \theta_3 + \\
& + \left(b_4 - \frac{1}{4}a_2 + \frac{1}{162}t_1t_3 + \frac{1}{243}t_2^2 - \frac{10}{3}c_3 + \frac{1}{30}R\right)\theta_1 \wedge \theta_4 + \\
& + \left(\frac{1}{27}t_2^2 + \frac{1}{18}t_1t_3 - \frac{1}{10}R + 2b_4 + \frac{3}{4}a_2\right)\theta_2 \wedge \theta_3 + \\
& + \left(\frac{2}{27}t_2t_3 + \frac{10}{3}c_2 + \frac{5}{12}b_5\right)\theta_2 \wedge \theta_4 + \left(\frac{1}{9}t_3^2 - \frac{5}{3}c_1 + \frac{1}{6}b_6\right)\theta_3 \wedge \theta_4 + \\
& + \frac{1}{12}(2t_{14} + t_{23} - 6t_2t_3 - 3t_{32})\theta^2 \wedge \theta^4 + \frac{1}{12}(4t_{24} - 9t_3^2 - 3t_{33})\theta^3 \wedge \theta^4, \\
d\Gamma_0 &= \Gamma_+ \wedge \Gamma_- - \frac{1}{4}b_1\theta_0 \wedge \theta_1 + \left(-\frac{1}{6}b_2 - \frac{1}{162}t_1^2 + \frac{5}{6}c_5\right)\theta_0 \wedge \theta_2 + \\
& + \left(-\frac{1}{54}t_1t_2 - \frac{1}{12}b_3 + \frac{1}{8}a_1\right)\theta_0 \wedge \theta_3 + \\
& - \left(\frac{1}{81}t_1t_3 + \frac{2}{243}t_2^2 + \frac{5}{6}c_3 + \frac{1}{60}R\right)\theta_0 \wedge \theta_4 +
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{1}{162} t_1 t_2 - \frac{20}{3} c_4 - \frac{1}{6} b_3 - \frac{3}{8} a_1 \right) \theta_1 \wedge \theta_2 + \\
& + \left( -\frac{1}{81} t_1 t_3 - \frac{2}{243} t_2^2 + \frac{20}{3} c_3 + \frac{1}{30} R \right) \theta_1 \wedge \theta_3 + \\
& + \left( -\frac{1}{18} t_2 t_3 - \frac{1}{8} a_3 + \frac{1}{12} b_5 \right) \theta_1 \wedge \theta_4 + \\
& + \left( \frac{1}{54} t_2 t_3 + \frac{3}{8} a_3 - \frac{20}{3} c_2 + \frac{1}{6} b_5 \right) \theta_2 \wedge \theta_3 + \\
& + \left( -\frac{1}{18} t_3^2 + \frac{5}{6} c_1 + \frac{1}{6} b_6 \right) \theta_2 \wedge \theta_4 + \frac{1}{4} b_7 \theta_3 \wedge \theta_4, \\
d\Gamma_1 = & -\frac{1}{8} b_1 \theta_0 \wedge \theta_1 - \frac{1}{8} b_2 \theta_0 \wedge \theta_2 - \frac{1}{8} (b_3 + a_1) \theta_0 \wedge \theta_3 - \frac{1}{8} (b_4 + a_2) \theta_0 \wedge \theta_4 + \\
& + \left( \frac{3}{8} a_1 - \frac{1}{4} b_3 \right) \theta_1 \wedge \theta_2 + \left( \frac{1}{4} a_2 - b_4 \right) \theta_1 \wedge \theta_3 - \frac{1}{8} (a_3 + b_5) \theta_1 \wedge \theta_4 + \\
& + \left( \frac{3}{8} a_3 - \frac{1}{4} b_5 \right) \theta_2 \wedge \theta_3 - \frac{1}{8} b_6 \theta_2 \wedge \theta_4 - \frac{1}{8} b_7 \theta_3 \wedge \theta_4,
\end{aligned}$$

where  $a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6, b_7, c_1, c_2, c_3, c_4, c_5$  and  $R$  are functions. All of these functions but  $R$  are determined by the differentials of torsions:

$$\begin{aligned}
dt_1 = & 2t_2\Gamma_- - 2t_1\Gamma_0 - 4t_1\Gamma_1 + \frac{3}{2}b_1\theta_0 + \left(2b_2 - \frac{4}{27}t_1^2 + 20c_5\right)\theta_1 + \\
& + \left(-\frac{4}{9}t_1t_2 - 60c_4 + 3b_3 - \frac{9}{2}a_1\right)\theta_2 + \\
& + \left(-\frac{4}{9}t_1t_3 - \frac{8}{27}t_2^2 + 60c_3 + 6b_4 - 9a_2\right)\theta_3 + \\
& + \left(-\frac{4}{9}t_2t_3 - \frac{9}{2}a_3 - 20c_2 + \frac{1}{2}b_5\right)\theta_4, \\
dt_2 = & 3t_3\Gamma_- + t_1\Gamma_+ - 4t_2\Gamma_1 + \left(\frac{1}{2}b_2 + \frac{2}{27}t_1^2 - 10c_5\right)\theta_0 + \\
& + \left(\frac{4}{27}t_1t_2 + 20c_4 + 2b_3 + \frac{9}{2}a_1\right)\theta_1 + 9(a_2 + b_4)\theta_2 + \\
& + \left(-\frac{4}{9}t_2t_3 + \frac{9}{2}a_3 - 20c_2 + 2b_5\right)\theta_3 + \left(-\frac{2}{3}t_3^2 + 10c_1 + \frac{1}{2}b_6\right)\theta_4, \\
dt_3 = & 2t_3\Gamma_0 + \frac{2}{3}t_2\Gamma_+ - 4t_3\Gamma_1 + \left(\frac{4}{81}t_1t_2 + \frac{20}{3}c_4 + \frac{1}{6}b_3 - \frac{3}{2}a_1\right)\theta_0 + \\
& + \left(\frac{4}{27}t_1t_3 + \frac{8}{81}t_2^2 - 20c_3 + 2b_4 - 3a_2\right)\theta_1 + \\
& + \left(\frac{4}{9}t_2t_3 - \frac{3}{2}a_3 + 20c_2 + b_5\right)\theta_2 + \left(\frac{4}{9}t_3^2 - \frac{20}{3}c_1 + \frac{2}{3}b_6\right)\theta_3 + \frac{1}{2}b_7\theta_4.
\end{aligned}$$

The function  $R$  is the Ricci scalar for the connection.

Before presenting the proof let us notice several facts and corollaries.

The theorem guarantees that every equivalence class of ODEs satisfying conditions (5.5) has its corresponding nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  geometry  $(M^5, [g, \Upsilon, A])$  with torsion in  $\Lambda_3$ . It may happen, however, that there are contact *non-equivalent* classes of ODEs defining *the same*  $\mathbf{GL}(2, \mathbb{R})$  geometries. (See also remark 5.9).

The Wünschmann conditions, although very complicated, possess nontrivial solutions. For example the equation

$$y^{(5)} = c \left( \frac{5y^{(3)3}(5 - 27cy''^2)}{9(1 + cy''^2)^2} + 10 \frac{y''y^{(3)}y^{(4)}}{1 + cy''^2} \right),$$

where  $c = \pm 1$  satisfies the Wünschmann conditions and is not contact equivalent to  $F = 0$ . Other examples are considered in section 6.

The connection of theorem 5.3 is a characteristic connection with torsion in  $\Lambda_3$ . If the Wünschmann ODE is general enough, the torsion may be quite arbitrary within  $\Lambda_3$ . From proposition 3.11 we know that the independent components of the curvature of a characteristic connection with  $T \in \Lambda_3$  are  $R, dA^{(3)}, dA^{(7)}$  and

$K^i$ . In the notation of theorem 5.3 they read:

$$\begin{aligned} dA^{(3)} &= \begin{pmatrix} 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & -3a_1 & -2a_2 & a_3 \\ 0 & 3a_1 & 0 & -3a_3 & 0 \\ -a_1 & 2a_2 & 3a_3 & 0 & 0 \\ -a_2 & -a_3 & 0 & 0 & 0 \end{pmatrix}, \\ dA^{(7)} &= \begin{pmatrix} 0 & b_1 & b_2 & b_3 & b_4 \\ -b_1 & 0 & 2b_3 & 8b_4 & b_5 \\ -b_2 & -2b_3 & 0 & 2b_5 & b_6 \\ -b_3 & -8b_4 & -2b_5 & 0 & b_7 \\ -b_4 & -b_5 & -b_6 & -b_7 & 0 \end{pmatrix}, \\ K &= \frac{\sqrt{3}}{3} (c_1 \ c_2 \ c_3 \ c_4 \ c_5)^T, \end{aligned}$$

and, as we said above, the Ricci scalar is given by the function  $R$ . The Ricci vector  $R_v^i = \Upsilon^{ijk} g_{jk}$  is as follows

$$R_v^i = \frac{7}{6}\sqrt{3} \left( t_3^2, -\frac{1}{3}t_2t_3, \frac{1}{9}t_1t_3 + \frac{2}{27}t_2^2, -\frac{1}{9}t_1t_2, \frac{1}{9}t_1^2 \right).$$

The Ricci tensor satisfies the following equations

$$\begin{aligned} R_{(ij)} &= \frac{1}{5}Rg_{ij} + \frac{2}{7}R_v^k \Upsilon_{ijk}, \\ dA^{(3)} &= 4R^{(3)}, \quad dA^{(7)} = \frac{2}{3}R^{(7)}. \end{aligned}$$

Using theorem 5.3 we can also express the Ricci tensor  $(Ric)^i_j = g^{ik} R_{kj}$  in terms of the endomorphisms  $E_-$ ,  $E_0$ ,  $E_+$ ,  $E_1$  of (2.5):

**Corollary 5.4.** *The Ricci tensor of a characteristic connection with torsion in  $\bigwedge_3$  has the following form in any adapted coframe*

$$\begin{aligned} Ric &= \left( \frac{1}{54}t_2^2 + \frac{1}{36}t_1t_3 - \frac{1}{20}R \right) E_1 + \frac{1}{8}b_1E_-^3 + \frac{1}{108}t_1^2E_-^2 + \\ &+ \left( -\frac{1}{54}t_1t_2 + \frac{1}{8}a_1 - \frac{1}{2}b_3 \right) E_- + \frac{5}{16}b_4E_0^3 + \left( \frac{1}{108}t_2^2 + \frac{1}{72}t_1t_3 \right) E_0^2 + \\ &+ \left( -\frac{17}{4}b_4 + \frac{1}{8}a_2 \right) E_0 - \frac{1}{8}b_7E_+^3 + \frac{1}{12}t_3^2E_+^2 + \left( -\frac{1}{8}a_3 + \frac{1}{2}b_5 - \frac{1}{18}t_2t_3 \right) E_+ + \\ &- \frac{5}{32}b_5E_0E_+E_0 + \frac{1}{8}b_6E_+E_0E_+ + \frac{1}{54}t_1t_2E_0E_- + \frac{5}{32}b_3E_0E_-E_0 + \\ &+ \frac{1}{8}b_2E_-E_0E_- - \frac{1}{18}t_2t_3E_0E_+. \end{aligned}$$

Of course, since the geometry is constructed from an ODE determined by the choice of  $F = F(x, y, y_1, y_2, y_3, y_4)$ , the coefficients  $a_1, \dots, a_3, b_1, \dots, b_7, R$  are expressible in terms of  $F$  and its derivatives. Given the connection of theorem 5.3 we calculated the explicit formulae for these coefficients and obtained the following

**Corollary 5.5.** *A  $\mathbf{GL}(2, \mathbb{R})$  geometry generated by a 5th order ODE satisfying Wünschmann conditions (5.5) has the following properties.*

*The torsion  $T$  vanishes iff*

$$F_{44} = 0.$$

*The 2-form  $dA^{(3)}$  vanishes iff*

$$\begin{aligned} &(\mathcal{D}F_4)_{34} - (\mathcal{D}F_3)_{44} - \frac{3}{5}(\mathcal{D}F_4)_4 F_{44} - \frac{4}{5}\mathcal{D}F_4 F_{444} + \frac{6}{25}F_{44}^2 F_4 + \frac{4}{25}F_4^2 F_{444} + \\ &+ \frac{3}{10}F_{34}F_{44} - \frac{1}{5}F_4 F_{344} + \frac{3}{5}F_3 F_{444} + F_{244} - \frac{1}{2}F_{433} = 0. \end{aligned}$$

*The 2-form  $dA^{(7)}$  vanishes iff*

$$F_{444} = 0.$$

*The Ricci vector  $R_v$  is aligned with the vector  $K$ , i.e.  $K = uR_v$ ,  $u \in \mathbb{R}$ , iff*

$$(\mathcal{D}F_4)_{44} - \frac{1}{2}F_{344} - \frac{2}{5}F_4 F_{444} - \frac{8}{15}F_{44}^2 + 7uF_{44}^2 = 0.$$

We skip writing the formula for the Ricci scalar since it is very complicated.

We now pass to the proof of theorem 5.3. On doing this we will apply a variant of the Cartan method of equivalence. This will be a rather long and complicated procedure. Thus, for the clarity of the presentation, we will divide the proof into three main steps, each of which will occupy its own respective section 5.1, 5.2 and 5.3. First, in section 5.1 we will prove lemma 5.6, which assures that a class of contact equivalent 5th order ODEs is a  $G$ -structure on a 4-order jet space  $J$ . Thus, we will have a bundle  $G \rightarrow J \times G \rightarrow J$ , a reduction of the frame bundle  $F(J)$ . In the second step, in section 5.2, we will use the Cartan method of equivalence in order to construct a submanifold  $P \subset J \times G$  together with a coframe on  $P$  which fulfills the requirements of proposition 4.1. This coframe, via proposition 4.1, will define an irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure for us and simultaneously will provide us with a  $\mathfrak{gl}(2, \mathbb{R})$  connection on the space of solutions of the ODE. The obstructions for an ODE to possess this structure, Wünschmann's expressions for  $F$ , will appear automatically in the course of the construction. This part of considerations is summarized in theorem 5.7. The  $\mathbf{GL}(2, \mathbb{R})$  structure obtained in this way will turn out to be nearly integrable, but the connection constructed will differ from the characteristic one. Therefore, in section 5.3, we will construct the characteristic connection associated with the  $\mathbf{GL}(2, \mathbb{R})$  structure obtained. This will have torsion in  $\bigwedge_3$ . This construction is described by lemma 5.8.

**5.1. 5th order ODE modulo contact transformations.** Let us consider a general 5th order ODE (5.1). We define the following coframe

$$\begin{aligned} \omega^0 &= dy - y_1 dx, \\ \omega^1 &= dy_1 - y_2 dx, \\ \omega^2 &= dy_2 - y_3 dx, \\ \omega^3 &= dy_3 - y_4 dx, \\ \omega^4 &= dy_4 - F(x, y, y_1, y_2, y_3, y_4) dx, \\ \omega_+ &= dx \end{aligned} \tag{5.6}$$

on  $J$ . We see that every solution of (5.1) is a curve  $c(x) = (x, y(x), y_1(x), y_2(x), y_3(x), y_4(x)) \subset J$  and the vector field  $\mathcal{D}$  on  $J$  has curves  $c(x)$  as the integral curves. The 1-forms  $(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4)$  annihilate  $\mathcal{D}$  whereas  $\mathcal{D} \lrcorner \omega_+ = 1$ . The 5-dimensional

space  $M^5$  of integral curves of  $\mathcal{D}$  is clearly the space of solutions of (5.1) and we have a fibration  $\mathbb{R} \rightarrow J \rightarrow M^5$ .

Suppose now, that equation (5.1) undergoes a contact transformation (5.3), which brings it to  $\bar{y}_5 = \bar{F}(\bar{x}, \bar{y}, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4)$ . Then the coframe transforms according to

$$(5.7) \quad \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega_+ \end{pmatrix} \mapsto \begin{pmatrix} \bar{\omega}^0 \\ \bar{\omega}^1 \\ \bar{\omega}^2 \\ \bar{\omega}^3 \\ \bar{\omega}^4 \\ \bar{\omega}_+ \end{pmatrix} = \begin{pmatrix} \alpha_0^0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_0^1 & \alpha_1^1 & 0 & 0 & 0 & 0 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 & 0 & 0 & 0 \\ \alpha_0^3 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & 0 & 0 \\ \alpha_0^4 & \alpha_1^4 & \alpha_2^4 & \alpha_3^4 & \alpha_4^4 & 0 \\ \alpha_0^5 & \alpha_1^5 & 0 & 0 & 0 & \alpha_5^5 \end{pmatrix} \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega_+ \end{pmatrix}.$$

Here  $\alpha_j^i$ ,  $i, j = 0, 1, 2, 3, 4, 5$ , are real functions on  $J$  defined by the formulae (5.3). They satisfy the nondegeneracy condition

$$\alpha_0^0 \alpha_1^1 \alpha_2^2 \alpha_3^3 \alpha_4^4 \alpha_5^5 \neq 0.$$

The transformed coframe encodes all the contact invariant information about the ODE. In particular, it preserves the simple ideal  $(\omega^0, \dots, \omega^4)$ , from which we can recover solutions of the transformed equation. Hence we have

**Lemma 5.6.** *A 5th order ODE  $y_5 = F(x, y, y_1, y_2, y_3, y_4)$  considered modulo contact transformations of variables is a  $G$ -structure on the 4-jet space  $J$ , such that the coframe  $(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4, \omega_+)$  of (5.6) belongs to it and the group  $G$  is given by the matrix in (5.7)*

**5.2. GL(2, ℝ) bundle over space of solutions.** Using the Cartan method we explicitly construct a submanifold  $P \subset J \times G$  and a coframe  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$  on  $P$  satisfying proposition 4.1. This part of the proof is divided into eight steps.

**1).** We observe that there is a natural choice for the forms  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$  of the coframe. Since we are going to build a GL(2, ℝ) structure on the space of solutions  $P$  must be a bundle over  $M^5$  and the forms  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$  must annihilate vectors tangent to leaves of the projection  $P \rightarrow M^5$ . But on  $J \times G$  there are six distinguished 1-forms given by

$$(5.8) \quad \begin{pmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta_+ \end{pmatrix} = \begin{pmatrix} \alpha_0^0 \omega^0 \\ \alpha_0^1 \omega^0 + \alpha_1^1 \omega^1 \\ \alpha_0^2 \omega^0 + \alpha_1^2 \omega^1 + \alpha_2^2 \omega^2 \\ \alpha_0^3 \omega^0 + \alpha_1^3 \omega^1 + \alpha_2^3 \omega^2 + \alpha_3^3 \omega^3 \\ \alpha_0^4 \omega^0 + \alpha_1^4 \omega^1 + \alpha_2^4 \omega^2 + \alpha_3^4 \omega^3 + \alpha_4^4 \omega^4 \\ \alpha_0^5 \omega_0 + \alpha_1^5 \omega_1 + \alpha_5^5 \omega_+ \end{pmatrix}.$$

These forms are the components of the canonical  $\mathbb{R}^6$  valued 1-form on  $J \times G$ . Five among these forms,  $\theta^0, \theta^1, \theta^2, \theta^3, \theta^4$  also annihilate vectors tangent to the projection  $J \times G \rightarrow M^5$ . We choose them to be the members of the sought coframe  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$ . Now we must construct a 9-dimensional submanifold  $P$  on which  $\theta^i$  satisfy equations (4.1) with some linearly independent forms  $\Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1$ .

2). We calculate  $d\theta^0$  and get

$$d\theta^0 = \left( \frac{d\alpha_0^0}{\alpha_0^0} - \frac{\alpha_0^1}{\alpha_1^1 \alpha_5^5} \theta_+ \right) \wedge \theta^0 + \frac{\alpha_0^0}{\alpha_1^1 \alpha_5^5} \theta_+ \wedge \theta^1 - \frac{\alpha_0^5}{\alpha_1^1 \alpha_5^5} \theta^0 \wedge \theta^1$$

For this equation to match (4.1) we define

$$(5.9) \quad \Gamma_+ = \theta_+$$

$$(5.10) \quad 4(\Gamma_1 + \Gamma_0) = \frac{d\alpha_0^0}{\alpha_0^0} - \frac{\alpha_0^1}{\alpha_1^1 \alpha_5^5} \theta_+ \mod \theta^i,$$

with yet unspecified  $\theta^i$  terms in (5.10), and set

$$(5.11) \quad \alpha_0^0 = -4\alpha_1^1 \alpha_5^5$$

to get  $-4$  coefficient in the  $\Gamma_+ \wedge \theta^1$  term. Thereby

$$d\theta^0 = 4(\Gamma_1 + \Gamma_0) \wedge \theta^0 - 4\Gamma_+ \wedge \theta^1 \mod \theta^i \wedge \theta^j$$

on the 23-dimensional subbundle of  $J \times G \rightarrow M^5$  given by (5.11). We see that the form  $\theta_+$  plays naturally the role of the connection 1-form  $\Gamma_+$ .

3). We calculate  $d\theta^1$  on the 23-dimensional bundle. In order to get

$$d\theta^1 = -\Gamma_- \wedge \theta^0 + (4\Gamma_1 + 2\Gamma_0) \wedge \theta^1 - 3\Gamma_+ \wedge \theta^2 \mod \theta^i \wedge \theta^j$$

we set

$$(5.12) \quad 4\Gamma_1 + 2\Gamma_0 = \frac{d\alpha_1^1}{\alpha_1^1} + \frac{\alpha_0^1 \alpha_2^2 - \alpha_1^1 \alpha_1^2}{\alpha_1^1 \alpha_2^2 \alpha_5^5} \theta_+ \mod \theta^i,$$

$$(5.13) \quad \begin{aligned} \Gamma_- = & -\frac{d\alpha_0^1}{4\alpha_1^1 \alpha_5^5} + \frac{\alpha_0^1 d\alpha_1^1}{4(\alpha_1^1)^2 \alpha_5^5} \\ & + \frac{(\alpha_0^1)^2 \alpha_2^2 + (\alpha_1^1)^2 \alpha_2^0 - \alpha_1^1 \alpha_2^1 \alpha_0^1}{4(\alpha_1^1)^2 \alpha_2^2 (\alpha_5^5)^2} \theta_+ \mod \theta^i, \end{aligned}$$

and

$$(5.14) \quad \alpha_2^2 = -\frac{\alpha_1^1}{3\alpha_5^5}$$

obtaining a 22-dimensional subbundle of  $J \times G \rightarrow M^5$  on which  $d\theta^0$  and  $d\theta^1$  are in the desired form.

4). At this point all four connection 1-forms  $\Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1$  are fixed up to the  $\theta^i$  terms. They are determined by the equations (5.9), (5.10), (5.12), (5.13). Thus we can not introduce any new 1-forms to bring  $d\theta^2$  into the desired form. Now to get  $d\theta^2$  in the form as in theorem 5.3, we may only use the yet unspecified coefficients  $\alpha$ s. That is why  $d\theta^2$  imposes more conditions on  $\alpha$ s. It follows that for  $d\theta^0, d\theta^1$  and  $d\theta^2$  to be of the form (4.1) the subbundle  $P$  must satisfy

$$\begin{aligned} \alpha_0^0 &= -4\alpha_1^1 \alpha_5^5, \\ \alpha_2^0 &= \frac{-75(\alpha_0^1)^2 + (\alpha_1^1)^2 (-20\mathcal{D}F_4 + 20F_3 + 7F_4^2)}{300\alpha_1^1 \alpha_5^5}, \\ \alpha_2^1 &= \frac{-15\alpha_0^1 + \alpha_1^1 F_4}{30\alpha_5^5}, \\ \alpha_2^2 &= -\frac{\alpha_1^1}{3\alpha_5^5}, \end{aligned}$$

$$\begin{aligned}
\alpha_0^3 &= [1800(\alpha_1^1 \alpha_5^5)^2]^{-1} \times \\
&[1125(\alpha_0^1)^3 + 45\alpha_0^1(\alpha_1^1)^2(20\mathcal{D}F_4 - 20F_3 - 7F_4^2) + \\
&2(\alpha_1^1)^3(100\mathcal{D}^2F_4 - 200F_2 - 30F_4\mathcal{D}F_4 - 60F_3F_4 - 11F_4^3)], \\
(5.15) \quad \alpha_1^3 &= \frac{225(\alpha_0^1)^2 - 30\alpha_0^1\alpha_1^1F_4 + (\alpha_1^1)^2(80\mathcal{D}F_4 - 100F_3 - 31F_4^2)}{1200\alpha_1^1(\alpha_5^5)^2}, \\
\alpha_2^3 &= \frac{5\alpha_0^1 - \alpha_1^1F_4}{20(\alpha_5^5)^2}, \\
\alpha_3^3 &= \frac{\alpha_1^1}{6(\alpha_5^5)^2}, \\
\alpha_1^4 &= [18000(\alpha_1^1)^2(\alpha_5^5)^3]^{-1} \times \\
&[-1125(\alpha_0^1)^3 + 225(\alpha_0^1)^2\alpha_1^1F_4 - 15\alpha_0^1(\alpha_1^1)^2(80\mathcal{D}F_4 - 100F_3 - 31F_4^2) + \\
&(\alpha_1^1)^3(-400\mathcal{D}^2F_4 + 1400F_2 + 240F_4\mathcal{D}F_4 + 180F_3F_4 + 11F_4^3)], \\
\alpha_2^4 &= \frac{-75(\alpha_0^1)^2 + 30\alpha_0^1\alpha_1^1F_4 + (\alpha_1^1)^2(-40\mathcal{D}F_4 + 80F_3 + 17F_4^2)}{600\alpha_1^1(\alpha_5^5)^3}, \\
\alpha_3^4 &= \frac{-5\alpha_0^1 + 3\alpha_1^1F_4}{30(\alpha_5^5)^3}, \\
\alpha_4^4 &= -\frac{\alpha_1^1}{6(\alpha_5^5)^3}.
\end{aligned}$$

The necessity of these conditions can be checked by a direct, quite lengthy calculations. We performed these calculations using the symbolic computation programs Maple and Mathematica.

We stress that conditions (5.15) are only *necessary* for  $d\theta^2$  to satisfy (4.1). It is because certain unwanted terms cannot be removed by *any* choice of subbundle  $P$ . Vanishing of these unwanted terms is a property of the ODE itself, and this is the reason for the Wünschmann conditions to appear.

More specifically, to achieve

$$d\theta^2 = -2\Gamma_- \wedge \theta^1 + 4\Gamma_1 \wedge \theta^2 - 2\Gamma_+ \wedge \theta^3 \quad \text{mod } \theta^i \wedge \theta^j$$

on the bundle defined by (5.11), (5.14) and (5.15) an ODE must satisfy

$$(5.16) \quad 50\mathcal{D}^2F_4 - 75\mathcal{D}F_3 + 50F_2 - 60F_4\mathcal{D}F_4 + 30F_3F_4 + 8F_4^3 = 0.$$

It follows from the construction that this condition, the first of (5.5), is invariant under the contact transformation of variables.

From now on we restrict our considerations only to contact equivalence class of ODEs satisfying (5.16). If (5.15) and (5.16) are satisfied then the three differentials  $d\theta^0$ ,  $d\theta^1$  and  $d\theta^2$  are precisely in the form (4.1).

**5).** The requirement that also  $d\theta^3$  is in the form (4.1) is equivalent to the following equation for  $\alpha_0^4$ :

$$\begin{aligned}
\alpha_0^4 &= [120000(\alpha_1^1 \alpha_5^5)^3]^{-1} \times \\
&[-1875(\alpha_0^1)^4 - 150(\alpha_0^1 \alpha_1^1)^2(20\mathcal{D}F_4 - 20F_3 - 7F_4^2) - \\
(5.17) \quad &40\alpha_0^1(\alpha_1^1)^3(50\mathcal{D}F_3 - 100F_2 + 30F_4\mathcal{D}F_4 - 40F_3F_4 - 9F_4^3) +
\end{aligned}$$

$$(\alpha^1_1)^4 \left( 400(-5\mathcal{D}^2 F_3 + 10\mathcal{D}F_2 - 6(\mathcal{D}F_4)^2 + 10F_3\mathcal{D}F_4 - 3F_3^2 + F_4\mathcal{D}F_3) + 120F_4^2(\mathcal{D}F_4 - 5F_3) - 63F_4^4 \right).$$

6). If condition (5.17) is also imposed we have

$$(\mathrm{d}\theta^4 + 4\Gamma_- \wedge \theta^3 - 4(\Gamma - \Gamma_0) \wedge \theta^4) \wedge \theta^0 \wedge \theta^1 = 0 \mod \theta^i.$$

However,

$$\mathrm{d}\theta^4 \wedge \theta^0 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0$$

if and only if second condition of (5.5) is satisfied:

$$(5.18) \quad \begin{aligned} & 375\mathcal{D}^2 F_3 - 1000\mathcal{D}F_2 + 350\mathcal{D}F_4^2 + 1250F_1 - 650F_3\mathcal{D}F_4 + 200F_3^2 - \\ & 150F_4\mathcal{D}F_3 + 200F_2F_4 - 140F_4^2\mathcal{D}F_4 + 130F_3F_4^2 + 14F_4^4 = 0. \end{aligned}$$

Again it follows from the construction that condition (5.18), considered simultaneously with (5.16), is invariant under contact transformations of the variables. From now on, we assume that all our 5th order ODEs (5.1) satisfy both conditions (5.16), (5.18). It follows that it is still not sufficient to force  $\mathrm{d}\theta^4$  to satisfy the system (4.1), since without further assumptions on  $F$ , we do *not* have  $\mathrm{d}\theta^4 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0$ . To achieve this it is necessary and sufficient to impose the last restriction on  $F$ :

$$(5.19) \quad \begin{aligned} & 1250\mathcal{D}^2 F_2 - 6250\mathcal{D}F_1 + 1750\mathcal{D}F_3\mathcal{D}F_4 - 2750F_2\mathcal{D}F_4 - \\ & 875F_3\mathcal{D}F_3 + 1250F_2F_3 - 500F_4\mathcal{D}F_2 + 700(\mathcal{D}F_4)^2 F_4 + \\ & 1250F_1F_4 - 1050F_3F_4\mathcal{D}F_4 + 350F_3^2 F_4 - 350F_4^2\mathcal{D}F_3 + \\ & 550F_2F_4^2 - 280F_4^3\mathcal{D}F_4 + 210F_3F_4^3 + 28F_4^5 + 18750F_y = 0. \end{aligned}$$

7). Assuming that  $F$  satisfies conditions (5.5) and fixing coefficients  $\alpha^i_j$  according to (5.15), (5.17) we are remained with a 11-dimensional subbundle of  $J \times G \rightarrow M^5$  parametrized by  $(x, y, y_1, y_2, y_3, y_4, \alpha^1_0, \alpha^1_1, \alpha^5_5, \alpha^5_0, \alpha^5_1)$ . It follows that the forms  $\Gamma_0, \Gamma_1, \Gamma_-, \Gamma_+$  on this bundle are

$$(5.20) \quad \begin{aligned} \Gamma_+ &= \theta_+, \\ \Gamma_0 &= \frac{\mathrm{d}\alpha^5_5}{2\alpha^5_5} - \frac{5\alpha^1_0 + \alpha^1_1 F_4}{20\alpha^1_1 \alpha^5_5} \theta_+ \mod \theta^i, \\ \Gamma_1 &= \frac{\mathrm{d}\alpha^1_1}{4\alpha^1_1} - \frac{\mathrm{d}\alpha^5_5}{4\alpha^5_5} + \frac{F_4}{20\alpha^5_5} \theta_+ \mod \theta^i, \\ \Gamma_- &= \frac{\mathrm{d}\alpha^1_0}{4\alpha^1_1 \alpha^5_5} - \frac{\alpha^1_0 \mathrm{d}\alpha^1_1}{4(\alpha^1_1)^2 \alpha^5_5} - \\ & \quad \frac{25(\alpha^1_0)^2 + 10\alpha^1_0 \alpha^1_1 F_4 + (\alpha^1_1)^2 (20\mathcal{D}F_4 - 20F_3 - 7F_4^2)}{400(\alpha^1_1 \alpha^5_5)^2} \theta_+ \mod \theta^i. \end{aligned}$$

8). In order to construct a 9-dimensional bundle and find the  $\theta^i$  terms in (5.20) we need to consider the  $\mathrm{d}\Gamma_A$  part of equations (4.1). Forcing  $\mathrm{d}\Gamma_A$  not to have  $\Gamma_A \wedge \theta^i$  terms we *uniquely* specify the  $\theta^i$  terms in (5.20). This requirement, in particular, fixes the coefficients  $\alpha^5_1$  and  $\alpha^5_0$  to be:

$$\alpha^5_1 = \frac{\alpha^5_5(10\mathcal{D}F_{44} + 5F_{34} + 6F_4F_{44})}{50},$$



$$(5.21) \quad \alpha_0^5 = \frac{\alpha_5^5}{250} [50(\mathcal{D}F_{34} + 7F_{24} - 5F_{33}) + 5F_4(6\mathcal{D}F_{44} - 37F_{34}) + 2F_{44}(-60\mathcal{D}F_4 + 145F_3 + 21F_4^2)].$$

Now all the forms  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1)$  are well defined and independent on a 9-dimensional manifold  $P$  parametrized by  $(y, y_1, y_2, y_3, y_4, x, \alpha_0^1, \alpha_1^1, \alpha_5^5)$ . We calculate structural equations (4.1) for these forms and have the following

**Theorem 5.7.** *A 5th order ODE  $y^{(5)} = F(x, y, y', y'', y^{(3)}, y^{(4)})$  considered modulo contact transformation of variables has an irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure on the space of its solution  $M^5$  together with a  $\mathfrak{gl}(2, \mathbb{R})$  connection  $\Gamma$  if and only if its defining function  $F = F(x, y, y_1, y_2, y_3, y_4)$  satisfies the contact invariant Wünschmann conditions (5.5). The bundle  $\mathbf{GL}(2, \mathbb{R}) \rightarrow P \rightarrow M^5$  is given by the equations (5.15), (5.17) and (5.21). The first structural equations for the connection  $\Gamma = (\Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1)$  on  $P$  read*

$$(5.22) \quad \begin{aligned} d\theta^0 &= 4(\Gamma_1 + \Gamma_0) \wedge \theta^0 - 4\Gamma_+ \wedge \theta^1 + t_1\theta^0 \wedge \theta^1 + t_2\theta^0 \wedge \theta^2 + t_3\theta^0 \wedge \theta^3, \\ d\theta^1 &= -\Gamma_- \wedge \theta^0 + (4\Gamma_1 + 2\Gamma_0) \wedge \theta^1 - 3\Gamma_+ \wedge \theta^2 + \frac{1}{2}t_1\theta^0 \wedge \theta^2 + \frac{1}{3}t_2\theta^0 \wedge \theta^3 + \frac{1}{4}t_3\theta^0 \wedge \theta^4 + t_2\theta^1 \wedge \theta^2 + t_3\theta^1 \wedge \theta^3, \\ d\theta^2 &= -2\Gamma_- \wedge \theta^1 + 4\Gamma_1 \wedge \theta^2 - 2\Gamma_+ \wedge \theta^3 + \frac{2}{9}t_1\theta^0 \wedge \theta^3 + \frac{1}{18}t_2\theta^0 \wedge \theta^4 + \frac{1}{3}t_1\theta^1 \wedge \theta^2 + \frac{8}{9}t_2\theta^1 \wedge \theta^3 + \frac{2}{3}t_3\theta^1 \wedge \theta^4 + t_3\theta^2 \wedge \theta^3, \\ d\theta^3 &= -3\Gamma_- \wedge \theta^2 + (4\Gamma_1 - 2\Gamma_0) \wedge \theta^3 - \Gamma_+ \wedge \theta^4 + \frac{1}{12}t_1\theta^0 \wedge \theta^4 + \frac{1}{3}t_1\theta^1 \wedge \theta^3 + \frac{1}{3}t_2\theta^1 \wedge \theta^4 + t_2\theta^2 \wedge \theta^3 + \frac{3}{2}t_3\theta^2 \wedge \theta^4, \\ d\theta^4 &= -4\Gamma_- \wedge \theta^3 + 4(\Gamma_1 - \Gamma_0) \wedge \theta^4 + \frac{1}{3}t_1\theta^1 \wedge \theta^4 + t_2\theta^2 \wedge \theta^4 + 3t_3\theta^3 \wedge \theta^4, \end{aligned}$$

with the torsion coefficients

$$\begin{aligned} t_3 &= \frac{6(\alpha_5^5)^2}{5\alpha_1^1} F_{44}, \\ t_2 &= \frac{9\alpha_5^5}{50(\alpha_1^1)^2} [\alpha_1^1(10\mathcal{D}F_{44} + 3F_4F_{44}) + 5\alpha_0^1F_{44}], \\ t_1 &= [1000(\alpha_1^1)^3]^{-1} \times \\ &\quad \left( 225(\alpha_0^1)^2F_{44} + 90\alpha_0^1\alpha_1^1(10\mathcal{D}F_{44} + 3F_4F_{44}) + \right. \\ &\quad \left. -9(\alpha_1^1)^2[20(5\mathcal{D}F_{34} + 20F_{24} - 15F_{33} + 3F_4\mathcal{D}F_{44} - 11F_4F_{34}) + \right. \\ &\quad \left. + F_{44}(-120\mathcal{D}F_4 + 340F_3 + 51F_4^2)] \right). \end{aligned}$$

Also the second structural equations are easily calculable but we skip them due to their complexity.

It is remarkable that the above  $\mathfrak{gl}(2, \mathbb{R})$  connection has torsion with not more than *three* functionally independent coefficients  $t_1, t_2, t_3$ . This suggests that the

$\mathbf{GL}(2, \mathbb{R})$  geometry on the 5-dimensional solution space  $M^5$  of the ODE is *nearly integrable* with torsion in the irreducible part  $\Lambda_3$  only. That it is really the case will be shown below.

**5.3. Characteristic connection with torsion in  $\Lambda_3$ .** As we know from section 3, given an irreducible  $\mathbf{GL}(2, \mathbb{R})$ -structure  $(M^5, [g, \Upsilon, A])$ , we can ask if such a structure is nearly integrable. According to propositions 3.5 and 3.6, the necessary and sufficient condition for nearly integrability is that the structure admits a  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection with *totally skew symmetric* torsion.

In our case of ODEs satisfying Wünschmann conditions we have a  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection of theorem 5.7, whose torsion is expressible in terms of three independent functions. This torsion, however, has quite complicated algebraic structure, in particular it is *not* totally skew symmetric.

It appears that an irreducible  $\mathbf{GL}(2, \mathbb{R})$  structure  $(M^5, [g, \Upsilon, A])$  associated with any 5th order ODE satisfying conditions (5.5) admits another  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection that *has* totally skew symmetric torsion. Thus all structures  $(M^5, [g, \Upsilon, A])$  originating from Wünschmann 5th order ODEs are nearly integrable; the new connection is their characteristic connection. Even more interesting is the fact that its torsion is still more special: it is always in  $\Lambda_3$ .

One way of seeing this is to calculate the Weyl connection  $\overset{w}{\Gamma}$  for the corresponding  $(M^5, [g, \Upsilon, A])$  and to decompose it according to (3.14). Here we prefer another method – the analysis in terms of the Cartan bundle  $P$  of theorem 5.7.

**Lemma 5.8.** *Consider a contact equivalence class of 5th order ODEs satisfying conditions (5.5). Let  $\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1$  and  $t_1, t_2, t_3$  be the objects of theorem 5.7. Then there is a  $\mathfrak{gl}(2, \mathbb{R})$  connection  $\tilde{\Gamma} = (\tilde{\Gamma}_+, \tilde{\Gamma}_-, \tilde{\Gamma}_0, \tilde{\Gamma}_1)$  whose torsion  $\tilde{T}_{jk}^i$  is totally skew symmetric and has its associated 3-form in  $\tilde{T} \in *\Lambda_3$ . Explicitly:*

$$\begin{aligned} \tilde{T} = & \frac{1}{12}t_1(-\theta^0 \wedge \theta^1 \wedge \theta^4 + 2\theta^0 \wedge \theta^2 \wedge \theta^3) + \\ & \frac{1}{12}t_2(-\theta^0 \wedge \theta^2 \wedge \theta^4 + 8\theta^1 \wedge \theta^2 \wedge \theta^3) + \\ & \frac{1}{4}t_3(-\theta^0 \wedge \theta^3 \wedge \theta^4 + 2\theta^1 \wedge \theta^2 \wedge \theta^4). \end{aligned}$$

*Proof.* Any  $\mathfrak{gl}(2, \mathbb{R})$  connection  $\tilde{\Gamma} = (\tilde{\Gamma}_+, \tilde{\Gamma}_-, \tilde{\Gamma}_0, \tilde{\Gamma}_1)$  compatible with the  $\mathbf{GL}(2, \mathbb{R})$  structure of theorem 5.7 is given by

$$(5.23) \quad \tilde{\Gamma}_A = \Gamma_A + \sum_i \gamma_{Ai} \theta^i, \quad A \in \{+, 0, -\}, \quad i = 0, \dots, 4,$$

$$\tilde{\Gamma}_1 = \Gamma_1$$

with arbitrary functions  $\gamma_{Ai}$ . We calculate structural equations  $d\theta + \tilde{\Gamma} \wedge \theta = \tilde{T}$  for  $\tilde{\Gamma}$  utilising equations (5.22), and ask if there exists a choice of  $\gamma_{Ai}$  such that the new torsion  $\tilde{T}_{jk}^i$  satisfies  $g_{il}\tilde{T}_{jk}^l = \tilde{T}_{[ijk]}$  and  $\tilde{T} = \frac{1}{6}g_{il}\tilde{T}_{jk}^l \theta^i \wedge \theta^j \wedge \theta^k \in *\Lambda_3$ . Using lemma 3.9 we easily find that the unique solution is given by

$$\begin{aligned} \tilde{\Gamma}_+ &= \Gamma_+ - \frac{1}{6}t_1\theta^0 - \frac{1}{3}t_2\theta^1 - \frac{1}{2}t_3\theta^2, \\ \tilde{\Gamma}_- &= \Gamma_- + \frac{1}{6}t_1\theta^2 + \frac{1}{3}t_2\theta^3 + \frac{1}{2}t_3\theta^4, \\ \tilde{\Gamma}_0 &= \Gamma_0 - \frac{1}{6}t_1\theta^1 - \frac{1}{3}t_2\theta^2 - \frac{1}{2}t_3\theta^3, \\ \tilde{\Gamma}_1 &= \Gamma_1, \end{aligned}$$

□

Lemma 5.8 together with the results of section 4 prove theorem 5.3.

*Remark 5.9.* Note that a passage from  $\Gamma_+$  to

$$\tilde{\Gamma}_+ = \Gamma_+ - \frac{1}{6}t_1\theta^0 - \frac{1}{3}t_2\theta^1 - \frac{1}{2}t_3\theta^2$$

belongs to a *larger* class of transformations than the contact transformations (5.7), (5.8); it involves a forbidden  $\theta^2$  term. Thus it may happen that there are nonequivalent classes of ODEs which define the same  $(M^5, [g, \Upsilon, A])$ . To distinguish between nonequivalent ODEs one has to use the connection of theorem 5.7.

## 6. EXAMPLES OF NEARLY INTEGRABLE $\mathbf{GL}(2, \mathbb{R})$ STRUCTURES FROM 5TH ORDER ODEs

In this section we provide examples of Wünschmann ODEs and nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  structures related to them. Since such structures have the torsions of their characteristic connections in  $\Lambda_3$ , then via proposition 3.11, they are characterized by the torsion  $T$ , the Ricci scalar  $R$ , the components of Maxwell 2-forms  $dA^{(3)}$ ,  $dA^{(7)}$ , and the vector  $K$ ; all these objects being associated to the characteristic connection  $\Gamma$ . There is also the unique Weyl connection  $\overset{w}{\Gamma}$  associated with these structures.

**6.1. Torsionfree structures.** We see from corollary 5.5 that

$$T \equiv 0 \quad \Longleftrightarrow \quad F_{44} \equiv 0.$$

Then  $\overset{w}{\Gamma} = \Gamma$  and all the curvature components but the Ricci scalar necessarily vanish. The following proposition can be checked by direct calculation.

**Proposition 6.1.** *The three nonequivalent differential equations*

$$y^{(5)} = c \left( \frac{5y^{(3)3}(5 - 27cy''^2)}{9(1 + cy''^2)^2} + 10 \frac{y''y^{(3)}y^{(4)}}{1 + cy''^2} \right),$$

with  $c = +1, 0, -1$ , represent the only three contact nonequivalent classes of 5th order ODEs having the corresponding nearly integrable  $\mathbf{GL}(2, \mathbb{R})$  structures  $(M^5, [g, \Upsilon, A])$  with the characteristic connection with vanishing torsion. In all three cases the holonomy of the Weyl connection  $\overset{w}{\Gamma}$  of structures  $(M^5, [g, \Upsilon, A])$  is reduced to the  $\mathbf{GL}(2, \mathbb{R})$ . For all the three cases the Maxwell 2-form  $dA \equiv 0$ . The corresponding Weyl structure is flat for  $c = 0$ . If  $c = \pm 1$ , then in the conformal class  $[g]$  there is an Einstein metric of positive ( $c = +1$ ) or negative ( $c = -1$ ) Ricci scalar. In case  $c = 1$  the manifold  $M^5$  can be identified with the homogeneous space  $\mathbf{SU}(1, 2)/\mathbf{SL}(2, \mathbb{R})$  with an Einstein  $g$  descending from the Killing form on  $\mathbf{SU}(1, 2)$ . Similarly in  $c = -1$  case the manifold  $M^5$  can be identified with the homogeneous space  $\mathbf{SL}(3, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})$  with an Einstein  $g$  descending from the Killing form on  $\mathbf{SL}(3, \mathbb{R})$ . In both cases with  $c \neq 0$  the metric  $g$  is not conformally flat.

**6.2. Structures with vanishing Maxwell form.** From now on we assume that

$$F_{44} \neq 0$$

and consider structures with vanishing Maxwell 2-form  $dA = 0$ . For such structures both torsion and curvature have at most 9 independent coefficients contained in  $T$ ,  $K$  and the scalar  $R$ . The simplest geometries in this class are those satisfying the additional equality

$$K^i = uR_{\nu}^i, \quad u \in \mathbb{R}.$$

Putting  $dA = 0$  and  $K^i = uR_v^i$  into structural equations of theorem 5.3 and using Bianchi identities we find that either

$$u = -\frac{1}{420}, \quad R = \frac{35}{54}(t_2^2 - 3t_1t_3)$$

or

$$u = \frac{2}{105}, \quad R = \frac{10}{27}(t_2^2 - 3t_1t_3).$$

Thus in these cases  $R$  is functionally dependent on  $t_1, t_2, t_3$  and the only invariants for such  $\mathbf{GL}(2, \mathbb{R})$  structures are  $u$  and the sign of  $R$ . For each possible values of  $u$  and  $\text{sgn}R$  we found a generating ODE.

**Proposition 6.2.** *Consider the equations*

$$(6.1) \quad F = \frac{5y_4^2}{3y_3} + \epsilon y_3^{5/3}, \quad \epsilon = -1, 0, 1,$$

$$(6.2) \quad F = \frac{5y_4^2}{4y_3},$$

and

$$(6.3) \quad F = \frac{5(8y_3^3 - 12y_2y_3y_4 + 3y_1y_4^2)}{6(2y_1y_3 - 3y_2^2)},$$

where the sign of expression  $(2y_1y_3 - 3y_2^2)$  is an invariant, and the singular locus  $2y_1y_3 - 3y_2^2 = 0$  separates nonequivalent equations with  $\pm$  signs. The equations generate all the six  $\mathbf{GL}(2, \mathbb{R})$  structures satisfying  $dA = 0$  and  $K^i = uR_v^i$ ,  $u \in \mathbb{R}$ .

$$\begin{aligned} \text{For (6.1)} \quad u &= -\frac{1}{420} \quad \text{and} \quad \text{sgn}R = \epsilon, \\ \text{for (6.2)} \quad u &= \frac{2}{105} \quad \text{and} \quad R = 0, \\ \text{for (6.3)} \quad u &= \frac{2}{105} \quad \text{and} \quad \text{sgn}R = \text{sgn}(3y_2^2 - 2y_1y_3). \end{aligned}$$

Moreover, the above ODEs can be also described in a geometric way by means of the symmetry group.

**Proposition 6.3.** *The equations (6.1), (6.2) and (6.3) are the only 5th order Wünschmann ODEs satisfying  $F_{44} \neq 0$ ,  $F_{444} = 0$  and possessing the maximal group of transitive contact symmetries of dimension greater than five. Equations  $F = \frac{5y_4^2}{3y_3}$  and  $F = \frac{5y_4^2}{4y_3}$  have 7-dimensional groups of symmetries, all the remaining have 6-dimensional ones.*

*Proof.* The proof is based on further application of the Cartan method of equivalence. Let us return to the coframe of theorem 5.7, which encodes all the contact invariant information about the ODE. If there are any nonconstant coefficients in the structural equations for this coframe we can use them for further reduction of the group  $\mathbf{GL}(2, \mathbb{R})$  and of the bundle  $P$ . For an ODE satisfying  $F_{44} \neq 0$  we normalize  $t_3 = 1$ ,  $t_2 = 0$ , which implies

$$\alpha^1_1 = \frac{6}{5}(\alpha^5_5)^2 F_{44}, \quad \alpha^1_0 = -\frac{6}{25}(\alpha^5_5)^2 (10\mathcal{D}F_{44} + 3F_4 F_{44}).$$

Now the coframe of theorem 5.7 is reduced to a 7-dimensional manifold  $P_7$  parameterized by  $(x, y, y_1, y_2, y_3, y_4, \alpha^5_5)$ , three 1-forms  $(\Gamma_0, \Gamma_-, \Gamma_+)$  become dependent on each other and we can use only one of them, our choice is  $\Gamma_0$ , to supplement  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_+)$  to an invariant coframe on  $P_7$ . Next we calculate structural

equations for the new coframe. The coefficients in these equations are built from  $\alpha^5_5$  and 16 functions  $f_1, \dots, f_{16}$  of  $x, y, y_1, \dots, y_4$ . In particular

$$\begin{aligned} d\theta^0 &= 6\Gamma_0 \wedge \theta^0 - 4\Gamma_+ \wedge \theta^1 + \frac{f_1}{(\alpha^5_5)^2} \theta^0 \wedge \theta^1 + \\ &+ \frac{f_2}{\alpha^5_5} \theta^0 \wedge \theta^2 + f_3 \theta^0 \wedge \theta^3 + f_4 \alpha^5_5 \theta^0 \wedge \theta^4, \end{aligned}$$

where for example

$$f_3 = -\frac{5F_{344}F_{44} + 10DF_{44}F_{444} + 6F_4F_{44}F_{444}}{F_{44}^3}, \quad f_4 = 5\frac{F_{444}}{F_{44}^2}.$$

Let us assume  $F_{444} = 0$  and consider two possibilities:  $f_3 \neq \text{const}$  and  $f_3 = \text{const}$ . If  $f_3 \neq \text{const}$  then it follows from the equations  $d^2\theta^i = 0$ ,  $d^2\Gamma_A = 0$  that  $f_2$  may not be a constant. Thus  $f_2/\alpha^5_5$  and  $f_3$  are *two* functionally independent coefficients in structural equations for the 7-dimensional coframe  $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_+, \Gamma_0)$ . According to the procedure of finding symmetries of ODEs, which is described in [19], the dimension of the group of contact symmetries of a corresponding 5-order ODE is not larger than the dimension of the coframe minus the number of the independent coefficients in the structural equations, that is  $7 - 2 = 5$ . It follows that ODEs possessing contact symmetry group greater than 5-dimensional necessarily satisfy  $f_3 = \text{const}$ . Let us assume  $f_3 = \text{const}$  then and we get from identities  $d^2\theta^i = 0$ ,  $d^2\Gamma_A = 0$  that (i) either  $f_3 = 2$  or  $f_3 = \frac{3}{2}$  and (ii) for both admissible values of  $f_3$  all the remaining nonvanishing functions  $f_j$  are expressible by  $f_1$ . For example, the system corresponding to  $f_3 = \frac{3}{2}$  is the following

$$\begin{aligned} d\theta^0 &= 6\Gamma_0 \wedge \theta^0 - 4\Gamma_+ \wedge \theta^1 + \frac{f_1}{(\alpha^5_5)^2} \theta^0 \wedge \theta^1 + \frac{3}{2} \theta^0 \wedge \theta^3 \\ d\theta^1 &= 4\Gamma_0 \wedge \theta^1 + \frac{2f_1}{7(\alpha^5_5)^2} \Gamma_+ \wedge \theta^0 - 3\Gamma_+ \wedge \theta^2 + \\ &\quad \frac{3f_1}{7(\alpha^5_5)^2} \theta^0 \wedge \theta^2 + \frac{3}{2} \theta^1 \wedge \theta^3 \\ d\theta^2 &= 2\Gamma_0 \wedge \theta^2 + \frac{4f_1}{7(\alpha^5_5)^2} \Gamma_+ \wedge \theta^1 - 2\Gamma_+ \wedge \theta^3 - \\ &\quad \frac{2f_1^2}{49(\alpha^5_5)^4} \theta^0 \wedge \theta^1 + \frac{4f_1}{21(\alpha^5_5)^2} \theta^0 \wedge \theta^3 + \\ &\quad \frac{f_1}{7(\alpha^5_5)^2} \theta^1 \wedge \theta^2 + \frac{1}{6} \theta^1 \wedge \theta^4 + \frac{3}{2} \theta^2 \wedge \theta^3 \\ d\theta^3 &= \frac{6f_1}{7(\alpha^5_5)^2} \Gamma_+ \wedge \theta^2 - \Gamma_+ \wedge \theta^4 - \frac{3f_1^2}{49(\alpha^5_5)^4} \theta^0 \wedge \theta^2 + \\ &\quad \frac{f_1}{14(\alpha^5_5)^2} \theta^0 \wedge \theta^4 + \frac{f_1}{7(\alpha^5_5)^2} \theta^1 \wedge \theta^3 + \frac{3}{4} \theta^2 \wedge \theta^4 \\ d\theta^4 &= -2\Gamma_0 \wedge \theta^4 + \frac{8f_1}{7(\alpha^5_5)^2} \Gamma_+ \wedge \theta^3 - \frac{4f_1^2}{49(\alpha^5_5)^4} \theta^0 \wedge \theta^3 + \\ &\quad \frac{f_1}{7(\alpha^5_5)^2} \theta^1 \wedge \theta^4 + \frac{3}{2} \theta^3 \wedge \theta^4 \\ d\Gamma_+ &= 2\Gamma_0 \wedge \Gamma_+ + \frac{3f_1^2}{98(\alpha^5_5)^4} \theta^0 \wedge \theta^1 + \frac{f_1}{14(\alpha^5_5)^2} \theta^0 \wedge \theta^3 + \frac{1}{8} \theta^1 \wedge \theta^4 \\ d\Gamma_0 &= \frac{f_1^2}{49(\alpha^5_5)^4} \Gamma_+ \wedge \theta^0 - \frac{1}{4} \Gamma_+ \wedge \theta^4 + \frac{3f_1^2}{196(\alpha^5_5)^4} \theta^0 \wedge \theta^2 + \\ &\quad \frac{f_1}{56(\alpha^5_5)^2} \theta^0 \wedge \theta^4 + \frac{f_1}{14(\alpha^5_5)^2} \theta^1 \wedge \theta^3 + \frac{3}{16} \theta^2 \wedge \theta^4. \end{aligned}$$

If  $f_1 = 0$  then to this system there corresponds a unique equivalence class of ODEs satisfying Wünschmann conditions and having 7-dimensional transitive contact symmetry group. The class is represented by

$$F = \frac{5y_4^2}{3y_3}.$$

In the case  $f_1 \neq 0$  we have next *two* nonequivalent classes of ODEs enumerated by the sign of  $f_1$  and possessing 6-dimensional transitive contact symmetry groups. Representatives of these classes are

$$F = \frac{5y_4^2}{3y_3} \pm y_3^{5/3},$$

where  $\pm 1 = \text{sgn} f_1$ .

In the similar vein we find that the only ODEs related to the case  $f_1 = 2$  are (6.2) and (6.3).  $\square$

**6.3. Simple structures with nonvanishing Maxwell form.** All the previous examples satisfy the contact invariant condition  $F_{444} = 0$ . In this paragraph we give examples of Wünschmann ODEs with  $F_{444} \neq 0$ . As such they will lead to the  $\mathbf{GL}(2, \mathbb{R})$  structures with the Maxwell form having a nonzero  $dA^{(7)}$  part. First and the simplest example of such equations is

$$(6.4) \quad F = (y_4)^{(5/4)}.$$

The  $\mathbf{GL}(2, \mathbb{R})$  structure associated with this ODE has the following properties

$$dA^{(3)} = 0, \quad R = 0, \quad K = \frac{2}{105} R_v.$$

It is then an example of a structure with nonvanishing  $dA$  belonging to the 7-dimensional irreducible representation.

Next example is the ODE given by the formula

$$(6.5) \quad F = \frac{1}{9(y_1^2 + y_2)^2} \left( 5w(y_1^6 + 3y_1^4 y_2 + 9y_1^2 y_2^2 - 9y_2^3 - 4y_1^3 y_3 + 12y_1 y_2 y_3 + 4y_3^2 - 3y_4(y_1^2 + y_2)) + \right. \\ \left. 45y_4(y_1^2 + y_2)(2y_1 y_2 + y_3) - 4y_1^9 - 18y_1^7 y_2 - 54y_1^5 y_2^2 - 90y_1^3 y_2^3 + 270y_1 y_2^4 + \right. \\ \left. 15y_1^6 y_3 + 45y_1^4 y_2 y_3 - 405y_1^2 y_2^2 y_3 + 45y_2^3 y_3 + 60y_1^3 y_3^2 - 180y_1 y_2 y_3^2 - 40y_3^3 \right),$$

where<sup>3</sup>

$$w^2 = y_1^6 + 3y_1^4 y_2 + 9y_1^2 y_2^2 - 9y_2^3 - 4y_1^3 y_3 + 12y_1 y_2 y_3 + 4y_3^2 - 3y_1^2 y_4 - 3y_2 y_4.$$

Torsion and curvature for the corresponding  $\mathbf{GL}(2, \mathbb{R})$  structure are complicated and are of general algebraic form. Both these examples have 6-dimensional transitive group of contact symmetries.

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<sup>3</sup>Note that  $w = 0$  also gives rise to  $F$  satisfying conditions (5.5). But since such  $F$  has only quadratic  $y_4$ -dependence it is equivalent to one of proposition 6.1. Actually the one with  $c < 0$ .

**6.4. A remarkable nonhomogeneous example.** Finally, we present an example of 5th order ODEs satisfying Wünschmann conditions (5.5), which are generic, in a sense that the function  $F$  representing it satisfies  $F_{444} \neq 0$ , but which have the corresponding group of transitive symmetries of dimension  $D < 6$ . We consider an ansatz in which function  $F$  depends in a special way on only two coordinates  $y_3$  and  $y_4$ . Explicitly:

$$(6.6) \quad F = (y_3)^{5/3} q\left(\frac{y_4^3}{y_3^3}\right),$$

where  $q = q(z)$  is a sufficiently differentiable real function of its argument

$$z = \frac{y_4^3}{y_3^3}.$$

It is remarkable that the above  $F$  satisfies *all* Wünschmann conditions provided that

- either  $q(z) = \frac{5}{3}z^{2/3}$
- or function  $q(z)$  satisfies the following second order ODE:

$$(6.7) \quad 90z^{4/3}(3q - 4z^{2/3})q'' - 54z^{4/3}q'^2 + 30z^{1/3}(6q - 5z^{2/3})q' - 25q = 0.$$

In the first case  $F = \frac{5}{3}\frac{y_4^2}{y_3}$ , and we recover function (6.1) with 7-dimensional group of symmetries. Note that one of the solutions of equation (6.7) is  $q = \frac{5}{4}z^{2/3}$ , which corresponds to  $F = \frac{5}{4}\frac{y_4^2}{y_3}$ . Thus also the other solution with seven symmetries, the solution (6.2), is covered by this ansatz.

We observe that if function  $q(z)$  satisfies

$$(6.8) \quad 25q - 60zq' + 27z^{4/3}q'^2 = 0,$$

then it *also* satisfies the reduction (6.7) of conditions (5.5). Equation (6.8) can be solved by first putting it in the form

$$q' = \frac{5(2z^{1/3} \pm \sqrt{(4z^{2/3} - 3q)})}{9z^{2/3}}$$

and then by integrating, according to the sign  $\pm 1$ . In the upper sign case the integration gives  $q$  in an implicit form:

$$\frac{(2z^{1/3} + \sqrt{(4z^{2/3} - 3q)})^{24}(2\sqrt{(4z^{2/3} - 3q)} - z^{1/3})^3}{(2\sqrt{(4z^{2/3} - 3q)} + z^{1/3})^3(5z^{2/3} - 4q)^3} = \text{const.}$$

In the lower sign case the implicit equation for  $q$  is:

$$\frac{(2z^{1/3} + \sqrt{(4z^{2/3} - 3q)})^{24}(2\sqrt{(4z^{2/3} - 3q)} - z^{1/3})^3(5z^{2/3} - 4q)^3}{(2\sqrt{(4z^{2/3} - 3q)} + z^{1/3})^3q^{24}} = \text{const.}$$

Inserting these  $qs$  into (6.6) we have a quite nontrivial Wünschmann ODE  $F = F_{\pm}$ . We close this section with a remark that other solutions to the second order ODE (6.7) also provide examples of 5th order Wünschmann ODEs.

## 7. HIGHER ORDER ODEs

All our considerations about  $\mathbf{GL}(2, \mathbb{R})$  structures associated with ODEs of 5th order can be repeated for other orders. This is due to the following well known fact generalizing proposition 5.1:

**Proposition 7.1.** *For every  $n \geq 4$ , the ordinary differential equation*

$$y^{(n)} = 0$$

*has  $\mathbf{GL}(2, \mathbb{R}) \times_{\rho_n} \mathbb{R}^n$  as its group of contact symmetries. Here  $\rho_n : \mathbf{GL}(2, \mathbb{R}) \rightarrow \mathbf{GL}(n, \mathbb{R})$  is the  $n$ -dimensional irreducible representation of  $\mathbf{GL}(2, \mathbb{R})$ .*

The representation  $\rho_n$ , at the level of Lie algebra  $\mathfrak{gl}(2, \mathbb{R})$ , is given in terms of the Lie algebra generators

$$E_+ = \begin{pmatrix} 0 & n-1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & n-2 & \dots & 0 & 0 & 0 \\ & & & \dots & & & \\ 0 & 0 & 0 & \dots & 3 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 3 & \dots & 0 & 0 & 0 \\ & & & \dots & & & \\ 0 & 0 & 0 & \dots & n-2 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & n-1 & 0 \end{pmatrix},$$

$$E_0 = \begin{pmatrix} 1-n & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 3-n & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 5-n & \dots & 0 & 0 & 0 \\ & & & \dots & & & \\ 0 & 0 & 0 & \dots & n-5 & 0 & 0 \\ & & & \dots & 0 & n-3 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & n-1 \end{pmatrix}, \quad E_1 = (1-n)\mathbf{1},$$

where  $\mathbf{1}$  is the  $n \times n$  identity matrix. In case of dimension  $n = 5$  these matrices coincide with (2.6). They also satisfy the same commutation relations

$$[E_0, E_+] = -2E_+, \quad [E_0, E_-] = 2E_-, \quad [E_+, E_-] = -E_0,$$

where the commutator in the  $\mathfrak{gl}(2, \mathbb{R}) = \text{Span}_{\mathbb{R}}(E_-, E_+, E_0, E_1) \subset \text{End}(\mathbb{R}^n)$  is the usual commutator of matrices.

Now, we consider a general  $n$ -th order ODE

$$(7.1) \quad y^{(n)} = F(x, y, y', y'', y^{(3)}, \dots, y^{(n-1)}),$$

and as before, to simplify the notation, we introduce coordinates  $x, y, y_1 = y', y_2 = y'', y_3 = y^{(3)}, \dots, y_{n-1} = y^{(n-1)}$  on the  $(n+1)$ -dimensional jet space  $J$ . Introducing the  $n$  contact forms

$$(7.2) \quad \begin{aligned} \omega^0 &= dy - y_1 dx, \\ \omega^1 &= dy_1 - y_2 dx, \\ &\vdots \\ \omega^i &= dy_i - y_{i+1} dx, \\ &\vdots \\ \omega^{n-2} &= dy_{n-2} - y_{n-1} dx, \\ \omega^{n-1} &= dy_{n-1} - F(x, y, y_1, y_2, \dots, y_{n-1}) dx \end{aligned}$$



and the additional 1-form

$$w_+ = dx,$$

we define a *contact transformation* to be a diffeomorphism  $\phi : J \rightarrow J$  which transforms the above  $n + 1$  one-forms via:

$$\begin{aligned}\phi^* \omega^i &= \sum_{k=0}^i \alpha^i_k \omega^k, \quad i = 0, 1, \dots, n-1, \\ \phi^* w_+ &= \alpha^n_0 \omega^0 + \alpha^n_1 \omega^1 + \alpha^n_n w_+.\end{aligned}$$

Here  $\alpha^i_j$  are functions on  $J$  such that  $\prod_{i=0}^n \alpha^i_i \neq 0$  at each point of  $J$ .

Therefore, as in the case of  $n = 5$ , the contact equivalence problem for the  $n$ th order ODEs (7.1) can be studied in terms of the invariant forms  $(\theta^0, \theta^1, \dots, \theta^{n-1}, \Gamma_+)$  defined by

$$\begin{aligned}(7.3) \quad \theta^i &= \sum_{k=0}^i \alpha^i_k \omega^k, \quad i = 0, 1, \dots, n-1, \\ \Gamma_+ &= \alpha^n_0 \omega^0 + \alpha^n_1 \omega^1 + \alpha^n_n w_+.\end{aligned}$$

These forms initially live on an  $\frac{n^2+3n+8}{2}$ -dimensional manifold  $G \rightarrow J \times G \rightarrow J$ , with the  $G$ -factor parametrized by  $\alpha^i_j$ , such that  $\prod_{i=0}^n \alpha^i_i \neq 0$ .

Introducing  $\mathfrak{gl}(2, \mathbb{R})$ -valued forms

$$(7.4) \quad \Gamma = \Gamma_- E_- + \Gamma_+ E_+ + \Gamma_0 E_0 + \Gamma_1 E_1,$$

where  $(\Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1)$  are 1-forms on  $J \times G$ , we can specialize to  $F \equiv 0$ , and reformulate proposition 7.1 to

**Proposition 7.2.** *If  $F \equiv 0$  then one can chose  $\frac{n(n+1)}{2}$  parameters  $\alpha^i_j$ , as functions of  $x, y, y_1, \dots, y_{n-1}$  and the remaining three  $\alpha$ s, say  $\alpha^{i_1}_{j_1}, \alpha^{i_2}_{j_2}, \alpha^{i_3}_{j_3}$ , so that the  $(n+4)$ -dimensional manifold  $P$  parametrized by  $(x, y, y_1, \dots, y_{n-1}, \alpha^{i_1}_{j_1}, \alpha^{i_2}_{j_2}, \alpha^{i_3}_{j_3})$  is locally the contact symmetry group,  $P \cong \mathbf{GL}(2, \mathbb{R}) \times_{\rho_n} \mathbb{R}^n$ , of equation  $y^{(n)} = 0$ . Forms (7.3), after restriction to  $P$ , can be supplemented by three additional 1-forms  $(\Gamma_-, \Gamma_0, \Gamma_1)$ , so that  $(\theta^0, \theta^1, \dots, \theta^{n-1}, \Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1)$  constitute a basis of the left invariant forms on the Lie group  $P$ . The choice of  $\alpha$ s and  $\Omega$ s is determined by the requirement that basis  $(\theta^0, \theta^1, \dots, \theta^{n-1}, \Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1)$  satisfies*

$$(7.5) \quad \begin{aligned}d\theta + \Gamma \wedge \theta &= 0, \\ d\Gamma + \Gamma \wedge \Gamma &= 0,\end{aligned}$$

where  $\theta = (\theta^0, \theta^1, \dots, \theta^{n-1})^T$  is a column  $n$ -vector, and  $\Gamma$  is given by (7.4).

The defining equations (7.5) of the left invariant basis, when written explicitly in terms of  $\theta$ 's and  $\Gamma$ s, read

$$\begin{aligned}d\theta^0 &= (n-1)(\Gamma_1 + \Gamma_0) \wedge \theta^0 + (1-n)\Gamma_+ \wedge \theta^1, \\ d\theta^1 &= -\Gamma_- \wedge \theta^0 + [(n-1)\Gamma_1 + (n-3)\Gamma_0] \wedge \theta^1 + (2-n)\Gamma_+ \wedge \theta^2, \\ &\vdots \\ d\theta^k &= -k\Gamma_- \wedge \theta^{k-1} + [(n-1)\Gamma_1 + (n-2k-1)\Gamma_0] \wedge \theta^k +\end{aligned}$$

$$\begin{aligned}
(7.6) \quad & +(1+k-n)\Gamma_+ \wedge \theta^{k+1}, \\
& \vdots \\
d\theta^{n-1} &= (1-n)\Gamma_- \wedge \theta^{n-2} + (n-1)(\Gamma_1 - \Gamma_0) \wedge \theta^{n-1}, \\
d\Gamma_+ &= 2\Gamma_0 \wedge \Gamma_+, \\
d\Gamma_- &= -2\Gamma_0 \wedge \Gamma_-, \\
d\Gamma_0 &= \Gamma_+ \wedge \Gamma_-, \\
d\Gamma_1 &= 0.
\end{aligned}$$

This system can be analyzed in the same spirit as system (4.1) of section 4. Thus, we first consider the distribution

$$\mathfrak{h} = \{X \in TP \text{ s.t. } X \lrcorner \theta^i = 0, \ i = 0, 1, 2, \dots, n-1\}$$

annihilating  $\theta$ .

Then the first  $n$  equations of the system (7.6) guarantee that forms  $(\theta^0, \theta^1, \theta^2, \dots, \theta^{n-1})$  satisfy the Fröbenius condition,

$$d\theta^i \wedge \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \dots \wedge \theta^{n-1} = 0, \quad \forall i = 0, 1, 2, \dots, n-1$$

and that, in turn, the distribution  $\mathfrak{h}$  is integrable. Thus manifold  $P$  is foliated by 4-dimensional leaves tangent to the distribution  $\mathfrak{h}$ . The space of leaves of this distribution  $P/\mathfrak{h}$  can be identified with the solution space  $M^n = P/\mathfrak{h}$  of equation  $y^{(n)} = 0$ . This in particular means, that all equations (7.5) can be interpreted respectively as the first and the second structure equations for a  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection  $\Gamma$  having vanishing torsion and vanishing curvature. This  $\mathfrak{gl}(2, \mathbb{R})$ -valued connection originates from a certain  $\mathbf{GL}(2, \mathbb{R})$  (conformal) structure on the solution space  $M^n$ .

To make this last statement more precise we have to invoke a few results from Hilbert's theory of algebraic invariants [12] adapted to our situation of ODEs.

**7.1. Results from Hilbert's theory of algebraic invariants.** First we ask if for a given order  $n \geq 4$  of an ODE (7.1) with  $F = 0$  there exists a bilinear form  $\tilde{g}$  on  $P$  of proposition 7.2 such that it projects to a nondegenerate conformal metric on  $M^n$ . This is answered, in a bit more general form, by applying the *reciprocity law of Hermite* (see [12], p. 60), and its corollaries, due to Hilbert (see [12], p. 60).

To adapt Hilbert's results to our paper we introduce a definition of an *invariant of degree  $q$* . Let  $\tilde{t}$  be a *totally symmetric* covariant tensor field of rank  $q$  defined on the group manifold  $P$  of proposition 7.2.

**Definition 7.3.** The tensor field  $\tilde{t}$  is called a  $\mathbf{GL}(2, \mathbb{R})$ -invariant of degree  $q$ , if and only if, it is degenerate on  $\mathfrak{h}$  and if for every  $X \in \mathfrak{h}$ , there exists a function  $c(X)$  on  $P$  such that

$$\mathcal{L}_X \tilde{t} = c(X) \tilde{t}.$$

The degeneracy condition means that  $\tilde{t}(X, \dots) = 0$ , for all  $X \in \mathfrak{h}$ .

In the following we will usually abbreviate the term 'a  $\mathbf{GL}(2, \mathbb{R})$ -invariant' to: 'an invariant'.

The first result from Hilbert's theory, adapted to our situation, is given by the following

**Proposition 7.4.** *For every  $n = 2m + 1$ ,  $m = 2, 3, \dots$  there exists a unique, up to a scale, invariant  $\tilde{g}$  of second degree on  $P$ . This invariant, a degenerate symmetric conformal bilinear form  $\tilde{g}$  of signature  $(m + 1, m, 0, 0, 0, 0)$  on  $P$ , satisfies*

$$\mathcal{L}_X \tilde{g} = 2(n - 1)(X \lrcorner \Gamma_1) \tilde{g},$$

for all  $X \in \mathfrak{h}$ .

In case of *even* orders  $n = 2m$ , Hilbert's theory gives the following

**Proposition 7.5.** *For  $n = 2m$  every GL(2, ℝ)-invariant has degree  $q \geq 4$ .*

Thus, if  $n = 2m$ , we do *not* have a conformal metric on the solution space  $M^n$ .

Returning to *odd* orders, we present the quadratic invariants  $\tilde{g}$ , of proposition 7.4, for  $n < 10$ :

$$(7.7) \quad \begin{aligned} {}^5\tilde{g} &= 3(\theta^2)^2 - 4\theta^1\theta^3 + \theta^0\theta^4, & \text{if } n = 5, \\ {}^7\tilde{g} &= -10(\theta^3)^2 + 15\theta^2\theta^4 - 6\theta^1\theta^5 + \theta^0\theta^6, & \text{if } n = 7, \\ {}^9\tilde{g} &= 35(\theta^4)^2 - 56\theta^3\theta^5 + 28\theta^2\theta^6 - 8\theta^1\theta^7 + \theta^0\theta^8, & \text{if } n = 9. \end{aligned}$$

These expressions can be generalized to higher (odd)  $ns$ . We have the following

**Proposition 7.6.** *If  $n = 2m + 1$  and  $m \geq 2$ , the invariant  $\tilde{g}$  of proposition 7.4 is given by:*

$$\tilde{g} = \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \theta^j \theta^{2m-j} + \frac{1}{2} (-1)^m \binom{2m}{m} (\theta^m)^2.$$

*Remark 7.7.* This proposition is also valid for  $m = 1$ . For such  $m$ , the value of  $n$  is  $n = 3$ , and we are in the regime of *third* order ODEs. Such ODEs were considered by Wünschmann [21]. Since  ${}^3\tilde{g} = \theta^0\theta^2 - (\theta^1)^2$  is the only invariant in this case, the counterpart of the bundle  $P$  of proposition 7.2 is a 10-dimensional bundle  $P \cong \mathbf{O}(2, 3)$ , the full conformal group in Lorentzian signature  $(1, 2)$ . The counterpart of system (7.5)/(7.6) is given by Maurer-Cartan equations for  $\mathbf{O}(3, 2)$ :

$$\begin{aligned} d\theta^0 &= 2(\Gamma_1 + \Gamma_0) \wedge \theta^0 - 2\Gamma_+ \wedge \theta^1, \\ d\theta^1 &= -\Gamma_- \wedge \theta^0 + 2\Gamma_1 \wedge \theta^1 - \Gamma_+ \wedge \theta^2, \\ d\theta^2 &= -2\Gamma_1 \wedge \theta^1 + (2\Gamma_1 - 2\Gamma_0) \wedge \theta^2, \\ d\Gamma_+ &= 2\Gamma_0 \wedge \Gamma_+ + \frac{1}{2}\Gamma_3 \wedge \theta^0 + \Gamma_4 \wedge \theta^1, \\ d\Gamma_- &= -2\Gamma_0 \wedge \Gamma_- + \Gamma_2 \wedge \theta^1 + \frac{1}{2}\Gamma_3 \wedge \theta^2, \\ d\Gamma_0 &= \Gamma_+ \wedge \Gamma_- - \frac{1}{2}\Gamma_2 \wedge \theta^0 + \frac{1}{2}\Gamma_4 \wedge \theta^2, \\ d\Gamma_1 &= -\frac{1}{2}\Gamma_2 \wedge \theta^0 - \frac{1}{2}\Gamma_3 \wedge \theta^1 - \frac{1}{2}\Gamma_4 \wedge \theta^2, \\ d\Gamma_2 &= -\Gamma_3 \wedge \Gamma_- + 2\Gamma_2 \wedge \Gamma_0 + 2\Gamma_2 \wedge \Gamma_1, \\ d\Gamma_3 &= -2\Gamma_2 \wedge \Gamma_+ - 2\Gamma_4 \wedge \Gamma_- + 2\Gamma_3 \wedge \Gamma_1, \\ d\Gamma_4 &= -2\Gamma_4 \wedge \Gamma_0 - \Gamma_3 \wedge \Gamma_+ + 2\Gamma_4 \wedge \Gamma_1. \end{aligned}$$

Here, apart from  $\theta^0, \theta^1, \theta^2$  and  $\Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1$  we have also left invariant forms  $\Gamma_2, \Gamma_3, \Gamma_4$ .

Now we pass to the invariants of degree  $q = 3$ . The question of their existence was again determined by Hilbert (see [12], p. 60), in terms of the *reciprocity law of Hermite*. In the language of our paper we have the following

**Proposition 7.8.** *An invariant of third degree  $\tilde{\Upsilon}$  exists on  $P$  if and only if*

$$n = 4\mu + 1, \quad \mu \in \mathbb{N}.$$

Hilbert's theory, [12], p. 60, implies also the following:

**Proposition 7.9.** *In low dimensions  $n = 4\mu + 1$ , the unique up to a scale cubic invariant is given by*

- $n = 5$ :

$${}^5\tilde{\Upsilon} = (\theta^2)^3 - 2\theta^1\theta^2\theta^3 + \theta^0(\theta^3)^2 - \theta^0\theta^2\theta^4 + (\theta^1)^2\theta^4$$

- $n = 9$ :

$$\begin{aligned} {}^9\tilde{\Upsilon} = & 15(\theta^4)^3 - 36\theta^3\theta^4\theta^5 + 24\theta^2(\theta^5)^2 + 24(\theta^3)^2\theta^6 - 22\theta^2\theta^4\theta^6 - \\ & 8\theta^1\theta^5\theta^6 + 3\theta^0(\theta^6)^2 - 8\theta^2\theta^3\theta^7 + 12\theta^1\theta^4\theta^7 - 4\theta^0\theta^5\theta^7 + \\ & 3(\theta^2)^2\theta^8 - 4\theta^1\theta^3\theta^8 + \theta^0\theta^4\theta^8. \end{aligned}$$

The rough statement about the even orders,  $n = 2m$ , described in proposition 7.5, can be again refined in terms of the reciprocity law of Hermite. Following Hilbert we have

**Proposition 7.10.** *If  $4 \leq n = 2m$  the lowest order invariant tensor  $\tilde{\Upsilon}$  on  $P$  has degree four. This is unique (up to a scale) only if  $n = 4, 6, 8, 12$ . If  $n = 10$  or  $n = 14$  we have two independent quartic invariants  $\tilde{\Upsilon}$ ; if  $n = 16, 18, 20$  we have three independent quartic invariants; and so on.*

**Proposition 7.11.** *In low dimensions  $n = 2m$ , the quartic invariant tensor  $\tilde{\Upsilon}$  on  $P$  is given by*

- $n = 4$ :

$${}^4\tilde{\Upsilon} = -3(\theta^1)^2(\theta^2)^2 + 4\theta^0(\theta^2)^3 + 4(\theta^1)^3\theta^3 - 6\theta^0\theta^1\theta^2\theta^3 + (\theta^0)^2(\theta^3)^2$$

- $n = 6$ :

$$\begin{aligned} {}^6\tilde{\Upsilon} = & -32(\theta^2)^2(\theta^3)^2 + 48\theta^1(\theta^3)^3 + 48(\theta^2)^3\theta^4 - 76\theta^1\theta^2\theta^3\theta^4 - \\ & 12\theta^0(\theta^3)^2\theta^4 + 9(\theta^1)^2(\theta^4)^2 + 16\theta^0\theta^2(\theta^4)^2 - 12\theta^1(\theta^2)^2\theta^5 + \\ & 16(\theta^1)^2\theta^3\theta^5 + 4\theta^0\theta^2\theta^3\theta^5 - 10\theta^0\theta^1\theta^4\theta^5 + (\theta^0)^2(\theta^5)^2. \end{aligned}$$

- $n = 8$ :

$$\begin{aligned} {}^8\tilde{\Upsilon} = & -375(\theta^3)^2(\theta^4)^2 + 600\theta^2(\theta^4)^3 + 600(\theta^3)^3\theta^5 - 990\theta^2\theta^3\theta^4\theta^5 - \\ & 240\theta^1(\theta^4)^2\theta^5 + 81(\theta^2)^2(\theta^5)^2 + 360\theta^1\theta^3(\theta^5)^2 - 240\theta^2(\theta^3)^2\theta^6 + 360(\theta^2)^2\theta^4\theta^6 + \\ & 50\theta^1\theta^3\theta^4\theta^6 + 40\theta^0(\theta^4)^2\theta^6 - 234\theta^1\theta^2\theta^5\theta^6 - 60\theta^0\theta^3\theta^5\theta^6 + 25(\theta^1)^2(\theta^6)^2 + \\ & 24\theta^0\theta^2(\theta^6)^2 + 40\theta^1(\theta^3)^2\theta^7 - 60\theta^1\theta^2\theta^4\theta^7 - 10\theta^0\theta^3\theta^4\theta^7 + 24(\theta^1)^2\theta^5\theta^7 + \\ & 18\theta^0\theta^2\theta^5\theta^7 - 14\theta^0\theta^1\theta^6\theta^7 + (\theta^0)^2(\theta^7)^2. \end{aligned}$$

Among the small dimensions  $n = 7$  is quite special, since here the next invariant linearly and *functionally* independent of the metric  $\tilde{g}$  has  $q = 4$ . We have the following

**Proposition 7.12.** *In dimension  $n = 7$ , the invariant of the lowest degree is the metric  ${}^7\tilde{g}$ . There are no invariants of degree  $q = 3$  and only two linearly*

independent, invariants of degree  $q = 4$ . The first of them is  ${}^7\tilde{g}^2$ . The second can be chosen to be

$$\begin{aligned} {}^7\tilde{\Upsilon} = & 160(\theta^3)^4 - 480\theta^2(\theta^3)^2\theta^4 + 1035(\theta^2)^2(\theta^4)^2 - 1080\theta^1\theta^3(\theta^4)^2 + 540\theta^0(\theta^4)^3 - \\ & 1080(\theta^2)^2\theta^3\theta^5 + 1920\theta^1(\theta^3)^2\theta^5 - 180\theta^1\theta^2\theta^4\theta^5 - 1080\theta^0\theta^3\theta^4\theta^5 - 288(\theta^1)^2(\theta^5)^2 + \\ & 540\theta^0\theta^2(\theta^5)^2 + 540(\theta^2)^3\theta^6 - 1080\theta^1\theta^2\theta^3\theta^6 + 400\theta^0(\theta^3)^2\theta^6 + 540(\theta^1)^2\theta^4\theta^6 - \\ & 330\theta^0\theta^2\theta^4\theta^6 - 84\theta^0\theta^1\theta^5\theta^6 + 7(\theta^0)^2(\theta^6)^2. \end{aligned}$$

**7.2. Stabilizers of the irreducible  $\mathbf{GL}(2, \mathbb{R})$  in dimensions  $n < 10$ .** In dimensions  $n \leq 10$  the  $\mathbf{GL}(2, \mathbb{R})$  invariant tensors of low order  $q \leq 4$  turn out to be sufficient to reduce the  $\mathbf{GL}(n, \mathbb{R})$  group to  $\mathbf{GL}(2, \mathbb{R})$  in its irreducible  $n$ -dimensional representation.

Given an invariant tensor

$$\tilde{t} = \frac{1}{q!} t_{i_1 i_2 \dots i_q} \theta^{i_1} \dots \theta^{i_q}$$

of degree  $q$  on  $P$  and a  $\mathbf{GL}(n, \mathbb{R})$ -valued function  $a = (a^i_j)$  on  $P$ , at every point  $p \in P$ , we have a  $\mathbf{GL}(n, \mathbb{R})$ -action

$$(a^i_j, \tilde{t}_{i_1 i_2 \dots i_q}) \mapsto (\rho_n(a) \tilde{t})_{j_1 j_2 \dots j_q} = a^{i_1}_{j_1} a^{i_2}_{j_2} \dots a^{i_q}_{j_q} \tilde{t}_{i_1 i_2 \dots i_q}.$$

A subgroup  $G_{\tilde{t}}$  of  $\mathbf{GL}(n, \mathbb{R})$  consisting of  $a = (a^i_j)$  such that

$$\rho_n(a) \tilde{t} = (\det a)^{q/n} \tilde{t},$$

is the stabilizer of  $\tilde{t}$  at  $p \in P$ . Since  $\tilde{t}$  is an invariant then, obviously  $\mathbf{GL}(2, \mathbb{R}) \subset G_{\tilde{t}}$ .

This leads to the following question: how many invariants is needed in dimension  $n$  so that its common stabilizer is *precisely*  $\mathbf{GL}(2, \mathbb{R})$  in its  $n$  dimensional irreducible representation?

Inspecting Hilbert's results we checked that in dimensions  $4 \leq n \leq 9$  we have

**Theorem 7.13.** *For each  $n = 4, 5, 6, 7, 8, 9$ , the full stabilizer group of the respective invariant tensor  ${}^n\tilde{\Upsilon}$  of propositions 7.9, 7.11, 7.12, is the group  $\mathbf{GL}(2, \mathbb{R})$  in the  $n$ -dimensional irreducible representation  $\rho_n$ . In particular, if  $n = 5, 7, 9$  these stabilizers are subgroups of the respective pseudohomothetic groups  $\mathbf{CO}(3, 2)$ ,  $\mathbf{CO}(4, 3)$  and  $\mathbf{CO}(5, 4)$ , each in its defining representation.*

Thus in each of these dimensions it is the lowest order *nonquadratic* invariant what is responsible for the full reduction from  $\mathbf{GL}(n, \mathbb{R})$  to  $\mathbf{GL}(2, \mathbb{R})$ .

*Remark 7.14.* In dimension  $n = 5$ , using (7.7) and proposition 7.9 we define a *conformal metric*  $[{}^5g_{ij}]$  represented by

$${}^5g_{ij} = \frac{1}{2} \frac{\partial^2}{\partial \theta^i \partial \theta^j} ({}^5\tilde{g}), \quad i, j = 0, 1, 2, 3, 4$$

and a *conformal symmetric tensor of third degree*  $[{}^5\Upsilon_{ijk}]$  represented by

$${}^5\Upsilon_{ijk} = -\frac{\sqrt{3}}{8} \frac{\partial^3}{\partial \theta^i \partial \theta^j \partial \theta^k} ({}^5\tilde{\Upsilon}), \quad i, j, k, l = 0, 1, 2, 3, 4.$$

The convenient factor  $-\frac{\sqrt{3}}{8}$  in the expression for  ${}^5\Upsilon_{ijk}$  was chosen so that the pair  $({}^5g_{ij}, {}^5\Upsilon_{ijk})$  satisfies Cartan's identities (i)-(iii) of section 2. This leads to the  $\mathbf{GL}(2, \mathbb{R})$  geometries in dimension 5 considered in sections 3-5.

*Remark 7.15.* In the next odd dimension situation is quite similar, but now we have a quartic invariant  ${}^7\tilde{\Upsilon}$ . Thus apart from the *conformal metric*  $[{}^7g_{ij}]$  represented by

$${}^7g_{ij} = \frac{1}{2} \frac{\partial^2}{\partial\theta^i \partial\theta^j} ({}^7\tilde{g}), \quad i, j = 0, 1, 2, 3, 4, 5, 6$$

we have a *conformal symmetric tensor of fourth degree*  $[{}^7\Upsilon_{ijkl}]$  represented by

$$(7.8) \quad {}^7\Upsilon_{ijkl} = \frac{1}{24} \frac{\partial^4}{\partial\theta^i \partial\theta^j \partial\theta^k \partial\theta^l} ({}^7\tilde{\Upsilon}), \quad i, j, k, l = 0, 1, 2, 3, 4, 5, 6.$$

Note that  ${}^7\tilde{\Upsilon}$  of proposition 7.12 was chosen in such a way that the fourth order  ${}^7\Upsilon_{ijkl}$  satisfied

$${}^7g^{ij} {}^7\Upsilon_{ijkl} = 0, \quad \text{where} \quad {}^7g^{ij} {}^7g_{jk} = \delta^i_k.$$

This choice of the fourth order invariant is nevertheless arbitrary, since we can always get another invariant of the fourth order by replacing  ${}^7\Upsilon$  with

$${}^7\Upsilon_{ijkl} = c_1 {}^7\tilde{\Upsilon}_{ijkl} + c_2 {}^7\tilde{g}_{(ij} {}^7\tilde{g}_{kl)}.$$

It is interesting to note that the choice

$$c_1 = \frac{2\sqrt{5}}{\sqrt{3147}}, \quad c_2 = \frac{34}{\sqrt{15735}}$$

applied to  ${}^7\tilde{\Upsilon}$ , leads, via formula like (7.8), to  ${}^7\tilde{\Upsilon}_{ijkl}$  satisfying Cartan-like identity:

$${}^7g^{ih} {}^7g^{ef} {}^7\tilde{\Upsilon}_{ie(jk} {}^7\tilde{\Upsilon}_{lm)fh} = {}^7g_{(jk} {}^7g_{lm)}$$

and

$${}^7g^{ij} {}^7\tilde{\Upsilon}_{ijkl} = \frac{3}{2} c_2 {}^7g_{kl}, \quad \text{where} \quad {}^7g^{ij} {}^7g_{jk} = \delta^i_k.$$

Note also that the above Cartan-like identities are preserved under the conformal transformation

$$({}^7g_{ij}, {}^7\tilde{\Upsilon}_{ijkl}) \mapsto ({}^7g'_{ij}, {}^5\tilde{\Upsilon}'_{ijkl}) = (e^{2\phi} {}^7g_{ij}, e^{4\phi} {}^7\tilde{\Upsilon}_{ijkl}),$$

where  $\phi \in \mathbb{R}$ .

Thus the  $\mathbf{GL}(2, \mathbb{R})$  geometries in dimension  $n = 7$  may be defined by a conformal class of pairs of tensors  $[{}^7g_{ij}, {}^7\tilde{\Upsilon}_{ijkl}]$  with the properties and transformations as above.

*Remark 7.16.* By analogy, in dimensions  $n = 4, 6, 8$ , the irreducible  $\mathbf{GL}(2, \mathbb{R})$  geometries may be described in terms of a conformal tensor  $[{}^n\Upsilon_{ijkl}]$  represented by

$${}^n\Upsilon_{ijkl} = \frac{1}{24} \frac{\partial^4}{\partial\theta^i \partial\theta^j \partial\theta^k \partial\theta^l} ({}^n\tilde{\Upsilon}), \quad i, j, k, l = 0, 1, 2, \dots, n-1,$$

and obtained in terms of the respective *quartic* invariants  ${}^n\tilde{\Upsilon}$  of proposition 7.11.

*Remark 7.17.* Dimension  $n = 9$  is similar to dimension  $n = 5$ . A periodicity with period *four* is a remarkable feature of Hilbert's theory of algebraic invariants [12], p. 60.

**7.3. Wünschmann conditions for the existence of  $\mathbf{GL}(2, \mathbb{R})$  geometries on the solution space of ODEs.** An invariant tensor  $\tilde{t}$ , by its very definition, has a property that it descends to a nondegenerate conformal tensor  $[t]$  on the solutions space  $M^n = P/\mathfrak{h}$  of the equation  $y^{(n)} = 0$ . In particular in dimensions  $4 \leq n \leq 9$  the conformal class  $[{}^n\Upsilon]$ , corresponding to invariant tensors  ${}^n\tilde{\Upsilon}$  reduces the structure group of  $M^n$  to  $\mathbf{GL}(2, \mathbb{R})$  defining an irreducible  $\mathbf{GL}(2, \mathbb{R})$  geometry there. We do not know how many invariant tensors are needed to achieve this reduction for  $n > 9$ , but it is obvious that for a given  $n$  this number is finite, say  $w_n$ . Thus for each  $n \geq 3$  we have a finite number of invariants  ${}^n\tilde{\Upsilon}_I$ ,  $I = 1, 2, \dots, w_n$ , which descend to the solution space  $M^n$  of the equation  $y^{(n)} = 0$  equipping it with a  $\mathbf{GL}(2, \mathbb{R})$  structure. It is important that each of the invariants  ${}^n\tilde{\Upsilon}_I$  has only *constant* coefficients when expressed in terms of the invariant coframe  $(\theta^0, \dots, \theta^{n-1})$  on  $P$  (see, for example, every  ${}^n\tilde{\Upsilon}$  of the preceding section).

Now, we return to a *general*  $n$ -th order ODE (7.1). Thus we now have a general function  $F(x, y, y', y'', y^{(3)}, \dots, y^{(n-1)})$ , which determines the contact forms  $(\omega^0, \omega^1, \dots, \omega^{n-1}, w_+)$  by (7.2). Corresponding to these forms we have the invariant forms  $(\theta^0, \dots, \theta^{n-1}, \Gamma_+)$  of (7.3), which live on bundle  $J \times G$  over  $J$ . We can now ask the following question (this generalizes to arbitrary  $n > 3$  the similar question of section 5): What shall we assume about  $F$  defining the contact equivalence class of ODEs (7.1) that there exists a  $(4 + n)$ -dimensional subbundle  $P$  of  $J \times G$  on which the forms  $(\theta^0, \dots, \theta^{n-1}, \Gamma_+)$  satisfy:

$$\begin{aligned}
 d\theta^0 &= (n-1)(\Gamma_1 + \Gamma_0) \wedge \theta^0 + (1-n)\Gamma_+ \wedge \theta^1 + \frac{1}{2}T_{ij}^0 \theta^i \wedge \theta^j, \\
 d\theta^1 &= -\Gamma_- \wedge \theta^0 + [(n-1)\Gamma_1 + (n-3)\Gamma_0] \wedge \theta^1 + \\
 &\quad + (2-n)\Gamma_+ \wedge \theta^2 + \frac{1}{2}T_{ij}^1 \theta^i \wedge \theta^j, \\
 &\vdots \\
 d\theta^k &= -k\Gamma_- \wedge \theta^{k-1} + [(n-1)\Gamma_1 + (n-2k-1)\Gamma_0] \wedge \theta^k + \\
 (7.9) \quad &\quad + (1+k-n)\Gamma_+ \wedge \theta^{k+1} + \frac{1}{2}T_{ij}^k \theta^i \wedge \theta^j, \\
 &\vdots \\
 d\theta^{n-1} &= (1-n)\Gamma_- \wedge \theta^{n-2} + (n-1)(\Gamma_1 - \Gamma_0) \wedge \theta^{n-1} + \frac{1}{2}T_{ij}^{n-1} \theta^i \wedge \theta^j, \\
 d\Gamma_+ &= 2\Gamma_0 \wedge \Gamma_+ + \frac{1}{2}R_{+ij} \theta^i \wedge \theta^j, \\
 d\Gamma_- &= -2\Gamma_0 \wedge \Gamma_- + \frac{1}{2}R_{-ij} \theta^i \wedge \theta^j, \\
 d\Gamma_0 &= \Gamma_+ \wedge \Gamma_- + \frac{1}{2}R_{0ij} \theta^i \wedge \theta^j, \\
 d\Gamma_1 &= \frac{1}{2}R_{ij} \theta^i \wedge \theta^j.
 \end{aligned}$$

As first observed by Wünschmann [21] and then successively used by Newman and collaborators [18] this question can be reformulated into a nicer one. To make this reformulation we repeat our arguments from section 7.1.

Suppose that we are able to satisfy system (7.9) by forms (7.3). Consider the distribution

$$\mathfrak{h} = \{X \in TP \text{ s.t. } X \lrcorner \theta^i = 0, i = 0, 1, 2, \dots, n-1\}$$

annihilating  $\theta$ s. Despite of the fact that system (7.9) involves *new terms*, when compared with system (7.6), they do not destroy the integrability of the distribution  $\mathfrak{h}$ ; the first  $n$  equations (7.9) still guarantee that  $\mathfrak{h}$  is *integrable*. Thus manifold  $P$  is foliated by 4-dimensional leaves tangent to the distribution  $\mathfrak{h}$ . The space of leaves of this distribution  $P/\mathfrak{h}$  can be identified with the solution space  $M^n = P/\mathfrak{h}$  of equation (7.1). Now, on manifold  $P$  of system (7.9), we define  $w_n$  tensors  ${}^n\tilde{\Upsilon}_I$ , which formally are given by *the same* formulae that defined the  $w_n$  invariants  ${}^n\tilde{\Upsilon}_I$  of the flat system (7.6) needed to get the full reduction to  $\mathbf{GL}(2, \mathbb{R})$ . So, when defining the present  ${}^n\tilde{\Upsilon}_I$ , we use the same formulae as for the  $y^{(n)} = 0$  case, replacing forms  $\theta$  of the flat case, with forms  $\theta$  satisfying system (7.9). It is now easy to verify that the question about the conditions on  $F$  to admit  $P$  with system (7.9) is equivalent to the requirement that *all*  $w_n$  tensors  ${}^n\tilde{\Upsilon}_I$  transform *conformally* when Lie transported along the leaves of distribution  $\mathfrak{h}$ . Infinitesimally this condition is equivalent to the existence of functions  $c_I(X)$  on  $P$  such that

$$\mathcal{L}_X({}^n\tilde{\Upsilon}_I) = c_I(X) {}^n\tilde{\Upsilon}_I,$$

$\forall X \in \mathfrak{h}$ , and  $\forall I = 1, 2, \dots, w_n$ . If this is satisfied then tensors  ${}^n\tilde{\Upsilon}_I$  descend to a conformal class of tensors  $[{}^n\Upsilon_1, {}^n\Upsilon_2, \dots, {}^n\Upsilon_{w_n}]$  on the solution space  $M^n$  defining a  $\mathbf{GL}(2, \mathbb{R})$  there.

We know that in dimension  $n = 5$  the conformal preservation of  ${}^5\tilde{g}$  and  ${}^5\tilde{\Upsilon}$  is equivalent to the requirement on function  $F = F(x, y, y_1, y_2, y_3, y_4)$  to satisfy Wünschmann conditions (5.5). The generalization of this fact to other low dimensions  $4 \leq n < 10$  is given by the following

**Theorem 7.18.** *Let  $M^n$  be the solution space of  $n$ th order ODE*

$$(7.10) \quad y^{(n)} = F(x, y, y', y'', y^{(3)}, \dots, y^{(n-1)}),$$

*with  $4 \leq n < 10$ , and let*

$$\mathcal{D} = \partial_x + y_1 \partial_y + y_2 \partial_{y_1} + \dots + y_{n-1} \partial_{y_{n-2}} + F \partial_{y_{n-1}}$$

*be the total derivative. The necessary conditions for a contact equivalence class of ODEs (7.10) to define a principal  $\mathbf{GL}(2, \mathbb{R})$ -bundle  $\mathbf{GL}(2, \mathbb{R}) \rightarrow P \rightarrow M^n$  with invariants forms  $(\theta^0, \dots, \theta^{n-1}, \Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1)$  satisfying system (7.9) is that the defining function  $F$  of (7.10) satisfies  $n-2$  Wünschmann conditions given below:*

- $n = 4$ :

$$4\mathcal{D}^2 F_3 - 8\mathcal{D}F_2 + 8F_1 - 6\mathcal{D}F_3 F_3 + 4F_2 F_3 + F_3^3 = 0,$$

$$160\mathcal{D}^2 F_2 - 640\mathcal{D}F_1 + 144(\mathcal{D}F_3)^2 - 352\mathcal{D}F_3 F_2 + 144F_2^2 - 80\mathcal{D}F_2 F_3 + 160F_1 F_3 - 72\mathcal{D}F_3 F_3^2 + 88F_2 F_3^2 + 9F_3^4 + 16000F_y = 0,$$

- $n = 5$ :

$$50\mathcal{D}^2 F_4 - 75\mathcal{D}F_3 + 50F_2 - 60F_4 \mathcal{D}F_4 + 30F_3 F_4 + 8F_4^3 = 0$$

$$375\mathcal{D}^2 F_3 - 1000\mathcal{D}F_2 + 350\mathcal{D}F_4^2 + 1250F_1 - 650F_3 \mathcal{D}F_4 + 200F_3^2 - 150F_4 \mathcal{D}F_3 + 200F_2 F_4 - 140F_4^2 \mathcal{D}F_4 + 130F_3 F_4^2 + 14F_4^4 = 0$$



$$\begin{aligned}
& 1250\mathcal{D}^2F_2 - 6250\mathcal{D}F_1 + 1750\mathcal{D}F_3\mathcal{D}F_4 - 2750F_2\mathcal{D}F_4 - \\
& 875F_3\mathcal{D}F_3 + 1250F_2F_3 - 500F_4\mathcal{D}F_2 + 700(\mathcal{D}F_4)^2F_4 + \\
& 1250F_1F_4 - 1050F_3F_4\mathcal{D}F_4 + 350F_3^2F_4 - 350F_4^2\mathcal{D}F_3 + \\
& 550F_2F_4^2 - 280F_4^3\mathcal{D}F_4 + 210F_3F_4^3 + 28F_4^5 + 18750F_y = 0.
\end{aligned}$$

•  $n = 6$ :

$$45\mathcal{D}^2F_5 - 54\mathcal{D}F_4 + 27F_3 - 45\mathcal{D}F_5F_5 + 18F_4F_5 + 5F_5^3$$

$$\begin{aligned}
& 945\mathcal{D}^2F_4 - 1890\mathcal{D}F_3 + 900(\mathcal{D}F_5)^2 + 1575F_2 - 1350\mathcal{D}F_5F_4 + 333F_4^2 - \\
& 315\mathcal{D}F_4F_5 + 315F_3F_5 - 300\mathcal{D}F_5F_5^2 + 225F_4F_5^2 + 25F_5^4 = 0
\end{aligned}$$

$$\begin{aligned}
& 2835\mathcal{D}^2F_3 - 9450\mathcal{D}F_2 + 4320\mathcal{D}F_4\mathcal{D}F_5 + 14175F_1 - 5130\mathcal{D}F_5F_3 - \\
& 1728\mathcal{D}F_4F_4 + 1863F_3F_4 - 945\mathcal{D}F_3F_5 + 1800(\mathcal{D}F_5)^2F_5 + 1575F_2F_5 - \\
& 2160\mathcal{D}F_5F_4F_5 + 576F_4^2F_5 - 720\mathcal{D}F_4F_5^2 + 855F_3F_5^2 - \\
& 600\mathcal{D}F_5F_5^3 + 360F_4F_5^3 + 50F_5^5 = 0
\end{aligned}$$

$$\begin{aligned}
& 14175\mathcal{D}^2F_2 - 85050\mathcal{D}F_1 + 6480(\mathcal{D}F_4)^2 + 16200\mathcal{D}F_3\mathcal{D}F_5 - \\
& 31050\mathcal{D}F_5F_2 - 9720\mathcal{D}F_4F_3 + 3645F_3^2 - 6480\mathcal{D}F_3F_4 + \\
& 5400\mathcal{D}F_5^2F_4 + 11475F_2F_4 - 4320\mathcal{D}F_5F_4^2 + 864F_4^3 - 4725\mathcal{D}F_2F_5 + \\
& 10800\mathcal{D}F_4\mathcal{D}F_5F_5 + 14175F_1F_5 - 10800\mathcal{D}F_5F_3F_5 - 6480\mathcal{D}F_4F_4F_5 + \\
& 5940F_3F_4F_5 - 2700\mathcal{D}F_3F_5^2 + 4500(\mathcal{D}F_5)^2F_5^2 + 5175F_2F_5^2 - \\
& 7200\mathcal{D}F_5F_4F_5^2 + 2340F_4^2F_5^2 - 1800\mathcal{D}F_4F_5^3 + 1800F_3F_5^3 - 1500\mathcal{D}F_5F_5^4 + \\
& 1050F_4F_5^4 + 125F_5^6 + 297675F_y = 0
\end{aligned}$$

•  $n = 7$ :

$$245\mathcal{D}^2F_6 - 245\mathcal{D}F_5 + 98F_4 - 210\mathcal{D}F_6F_6 + 70F_5F_6 + 20F_6^3 = 0$$

$$\begin{aligned}
& 6860\mathcal{D}^2F_5 - 10976\mathcal{D}F_4 + 6615(\mathcal{D}F_6)^2 + 6860F_3 - 8330\mathcal{D}F_6F_5 + \\
& 1715F_5^2 - 1960\mathcal{D}F_5F_6 + 1568F_4F_6 - 1890\mathcal{D}F_6F_6^2 + 1190F_5F_6^2 + 135F_6^4 = 0
\end{aligned}$$

$$\begin{aligned}
& 9604\mathcal{D}^2F_4 - 24010\mathcal{D}F_3 + 15435\mathcal{D}F_5\mathcal{D}F_6 + 24010F_2 - 14749\mathcal{D}F_6F_4 - \\
& 5145\mathcal{D}F_5F_5 + 4459F_4F_5 - 2744\mathcal{D}F_4F_6 + 6615(\mathcal{D}F_6)^2F_6 + 3430F_3F_6 - \\
& 6615\mathcal{D}F_6F_5F_6 + 1470F_5^2F_6 - 2205\mathcal{D}F_5F_6^2 + 2107F_4F_6^2 - \\
& 1890\mathcal{D}F_6F_6^3 + 945F_5F_6^3 + 135F_6^5 = 0
\end{aligned}$$

$$\begin{aligned}
& 336140\mathcal{D}^2F_3 - 1344560\mathcal{D}F_2 + 180075(\mathcal{D}F_5)^2 + 432180\mathcal{D}F_4\mathcal{D}F_6 + \\
& 2352980F_1 - 624260\mathcal{D}F_6F_3 - 216090\mathcal{D}F_5F_4 + 64827F_4^2 - \\
& 144060\mathcal{D}F_4F_5 + 154350(\mathcal{D}F_6)^2F_5 + 192080F_3F_5 - 102900\mathcal{D}F_6F_5^2 + \\
& 17150F_5^3 - 96040\mathcal{D}F_3F_6 + 308700\mathcal{D}F_5\mathcal{D}F_6F_6 + 192080F_2F_6 -
\end{aligned}$$

$$246960\mathcal{D}F_6F_4F_6 - 154350\mathcal{D}F_5F_5F_6 + 113190F_4F_5F_6 - 61740\mathcal{D}F_4F_6^2 + \\ 132300(\mathcal{D}F_6)^2F_6^2 + 89180F_3F_6^2 - 176400\mathcal{D}F_6F_5F_6^2 + 47775F_5^2F_6^2 - \\ 44100\mathcal{D}F_5F_6^3 + 35280F_4F_6^3 - 37800\mathcal{D}F_6F_6^4 + 22050F_5F_6^4 + 2700F_6^6 = 0$$

$$2352980\mathcal{D}^2F_2 - 16470860\mathcal{D}F_1 + 1512630\mathcal{D}F_4\mathcal{D}F_5 + 2268945\mathcal{D}F_3\mathcal{D}F_6 - \\ 5126135\mathcal{D}F_6F_2 - 1512630\mathcal{D}F_5F_3 - 907578\mathcal{D}F_4F_4 + 648270(\mathcal{D}F_6)^2F_4 + \\ 907578F_3F_4 - 756315\mathcal{D}F_3F_5 + 1080450\mathcal{D}F_5\mathcal{D}F_6F_5 + 1596665F_2F_5 - \\ 1080450\mathcal{D}F_6F_4F_5 - 360150\mathcal{D}F_5F_5^2 + 288120F_4F_5^2 - 672280\mathcal{D}F_2F_6 + \\ 540225(\mathcal{D}F_5)^2F_6 + 1296540\mathcal{D}F_4\mathcal{D}F_6F_6 + 2352980F_1F_6 - \\ 1620675\mathcal{D}F_6F_3F_6 - 864360\mathcal{D}F_5F_4F_6 + 324135F_4^2F_6 - 648270\mathcal{D}F_4F_5F_6 + \\ 926100(\mathcal{D}F_6)^2F_5F_6 + 756315F_3F_5F_6 - 771750\mathcal{D}F_6F_5^2F_6 + 154350F_5^3F_6 - \\ 324135\mathcal{D}F_3F_6^2 + 926100\mathcal{D}F_5\mathcal{D}F_6F_6^2 + 732305F_2F_6^2 - 926100\mathcal{D}F_6F_4F_6^2 - \\ 617400\mathcal{D}F_5F_5F_6^2 + 524790F_4F_5F_6^2 - 185220\mathcal{D}F_4F_6^3 + 396900(\mathcal{D}F_6)^2F_6^3 + \\ 231525F_3F_6^3 - 661500\mathcal{D}F_6F_5F_6^3 + 209475F_5^2F_6^3 - 132300\mathcal{D}F_5F_6^4 + \\ 119070F_4F_6^4 - 113400\mathcal{D}F_6F_6^5 + 75600F_5F_6^5 + 8100F_6^7 + 65883440F_y = 0.$$

*Remark 7.19.* Although we calculated the Wünschmann conditions for  $n = 8$  and  $n = 9$ , we do not present them here due to their length. We remark, however that in any order  $n \geq 4$ , the  $n-2$  Wünschmann conditions, which by the very definition are conditions needed for an ODE to define a  $\mathbf{GL}(2, \mathbb{R})$  geometry on its solution space, are always of the *third* order in the derivatives of the function  $F$  which defines an ODE. In this sense they differ from the generalizations of Wünschmann conditions obtained by [7] and [8].

*Remark 7.20.* If  $n = 3$  we have only one Wünschmann condition [6, 21]:

$$9\mathcal{D}^2F_2 - 27\mathcal{D}F_1 - 18\mathcal{D}F_2F_2 + 18F_1F_2 + 4F_2^3 + 54F_y = 0.$$

and, if it satisfied, a conformal Lorentzian geometry associated with a metric

$${}^3g = \theta^0\theta^2 - (\theta^1)^2$$

is naturally defined on the solution space.

*Remark 7.21.* If  $n = 4$  the ODEs satisfying the two Wünschmann conditions lead to very *nontrivial* geometries on 4-dimensional solution spaces. These are a sort of conformal Weyl geometries, which instead of a metric are define in terms of the conformal rank four tensor  ${}^4\Upsilon$ . These geometries define a characteristic connection, which is  $\mathfrak{gl}(2, \mathbb{R})$  valued and has an exotic holonomy [2]. By this we mean that the holonomy of this *nonmetric* but *torsionless* connection does not appear on the Berger's list [2]. See also our account on this subject in [17].

*Remark 7.22.* Our studies of the ODES with  $n = 3, 4, 5$ , and the preliminary results about the cases with  $n \geq 7$ , make us to conjecture that if  $n \geq 7$  then the  $n-2$  Wünschmann conditions are too stringent to admit many solutions for  $F$ . Thus, we strongly believe, that if  $n \geq 7$  the corresponding  $\mathbf{GL}(2, \mathbb{R})$  geometries on the solution spaces of the Wünschmann ODEs are very special, such that, for example, their characteristic connections have identically vanishing curvatures. We intend to discuss these matters in a subsequent paper.

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